AN ANALYTIC APPROACH TO TURAEV'S SHADOW INVARIANT

Atle Hahn[1]

Grupo de Física Matemática da Universidade de Lisboa
Av. Prof. Gama Pinto, 2
PT-1649-003 Lisboa, Portugal

ABSTRACT. In the present paper we extend the "torus gauge fixing" approach by Blau and Thompson, which was developed in [10] for the study of Chern-Simons models with base manifolds M of the form $M = \Sigma \times S^1$, in a suitable way. We arrive at a heuristic path integral formula for the Wilson loop observables associated to general links in M. We then show that the right-hand side of this formula can be evaluated explicitly in a non-perturbative way and that this evaluation naturally leads to the face models in terms of which Turaev's shadow invariant is defined.

Key words: Chern-Simons models, Quantum invariants, White noise analysis

AMS subject classifications: 57M27, 60H40, 81T08, 81T45

1. INTRODUCTION

The study of the heuristic Chern-Simons path integral functional in [42] inspired the following two general approaches to quantum topology:

(A1) The perturbative approach based on the Chern-Simons path integral in the Landau gauge (cf. [20, 9, 8, 6, 14, 7, 5]).

(A2) The non-perturbative "quantum group approach" that comes in two different versions: the "surgery" version (cf. [35, 36] and the first part of [38]) and the "state sum" or "shadow" version (cf. [40, 39] and the second part of [38]).

While for the first approach the relationship to the Chern-Simons path integral is obvious it is not fully understood yet how the expressions that appear in the second approach are related to the Chern-Simons path integral. In other words the following problem has so far remained open (cf., e.g., [18]):

(P1) Derive the algebraic expressions (in particular, the R-matrices resp. the quantum 6j-symbols) that appear in approach (A2) directly from the Chern-Simons path integral.

Approach (A2) is considerably less complicated than approach (A1). Thus it is reasonable to expect that if one can solve problem (P1) then the corresponding path integral derivation will be less complicated than the path integral derivation[2] given in [20, 9, 8, 6, 14, 7, 5]. One could therefore hope that after having solved problem (P1) one can make progress towards the solution of the following problem, which can be considered to be one of the major open problems in quantum topology (cf. [32]):

(P2) Make rigorous sense of the heuristic path integral expressions for the Wilson loop observables (WLOs) that were studied in [42] (cf. formula (3.1) below).

As a first step towards the solution of problem (P2) one can try to solve the following weakened version:

(P2)' Make rigorous sense either of the original path integral expressions for the Wilson loop observables or, alternatively, of those path integral expressions that arise from the original ones after one has fixed a suitable gauge.

---

[1]Email: atle.hahn@gmx.de

[2]we remark that while the final perturbation series appearing in approach (A1) is rigorous (cf. [5]) the path integral expressions that are used for its derivation are not

The aim of the present paper is to give a partial solution of problems (P1) and (P2)'. In order to do so we will concentrate on the special situation where the base manifold M of the Chern-Simons model is of the form $M = \Sigma \times S^1$ and then apply the so-called "torus gauge fixing" procedure which was successfully used in [10] for the computation of the partition function of Chern-Simons models on such manifolds (cf. eq. (7.1) in [10]) and for the computation of the Wilson loop observables of a special type of links in M, namely links L that consist of "vertical" loops (cf. eq. (7.24) in [10], see also our Subsec. 6.2). The first question which we study in the present paper is the question whether is is possible to generalize the formulae (7.1) and (7.24) in [10] to general links L in M. The answer to this questions turns out to be "yes", cf. Eq. (3.31) below.

Next we study the question whether it is possible to give a rigorous meaning to the heuristic path integral expressions on the right-hand side of Eq. (3.31). Fortunately, it is very likely that also this question has a positive answer (cf. [26] and point (4) in Subsec. 7.2). In fact, due to the remarkable property of Eq. (3.31) that all the heuristic measures that appear there are of "Gaussian type" we can apply similar techniques as in the axial gauge approach to Chern-Simons models on $\mathbb{R}^3$ developed in [19, 4, 21, 22]. In particular, we can make use of white noise analysis and of the two regularization techniques "loop smearing" and "framing".

Finally, we study the question if and how the right-hand side of Eq. (3.31) can be evaluated explicitly and if, by doing this, one arrives at the same algebraic expressions for the corresponding quantum invariants as in the shadow version of approach (A2). It turns out that also this question has a positive answer, at least in all the special cases that we will study in detail.

The present paper is organized as follows. In Sec. 2 we summarize the relevant definitions and results from [10, 12, 23, 24] on Blau and Thompson's torus gauge fixing procedure. In Secs 3.1–3.3, we then apply the torus gauge fixing procedure to Chern-Simons models with compact base manifolds of the form $M = \Sigma \times S^1$. After introducing a suitable decomposition $\mathcal{A}^\perp = \hat{\mathcal{A}}^\perp \oplus \mathcal{A}_c^\perp$ in Subsec. 3.4 we finally arrive in Subsec. 3.5 at the aforementioned heuristic path integral formula (3.31) for the WLOs.

The rest of the paper is concerned with the question how one can make rigorous sense of (the inner integral in) the heuristic formula (3.31) and how one can evaluate its right-hand side explicitly. We proceed in three steps. In Sec. 4 (Step 1) we briefly summarize the rigorous realization of the integral functional $\Phi_B^\perp$ found in [23] and we then show in Sec. 5 how the whole inner integral can be realized rigorously and evaluated explicitly (Step 2). In Sec. 6 we then evaluate the whole right-hand side of formula (3.31) (Step 3) in several special cases. First we consider the case where the group G is Abelian (cf. Subsec. 6.1). Next we consider the case where $G = SU(2)$ and where the link L consists exclusively of "vertical" loops (this case was already studied successfully in Sec. 7.6 in [10]). In Subsec. 6.3 we then study the case where $G = SU(2)$ and where the link L has no double points, and demonstrate how in this situation the face models by which the shadow invariant is defined arise naturally. Finally, in Sec. 7 we comment on the results to be expected when completing the computations for the WLOs of general links, which we plan to carry out in the near future, and we then give a list of suggestions for additional generalizations/extensions of the results of the present paper.

Convention: In the present paper, the symbol "$\sim$" will denote "equality up to a multiplicative constant". Sometimes we allow this multiplicative "constant" to depend on the "charge" k of the model, but it will never depend on the link L which we will fix in Subsec. 3.1 below.

2. Torus gauge fixing for manifolds $M = \Sigma \times S^1$

Let M be a smooth manifold of the form $M = \Sigma \times S$ where $S \in \{S^1, \mathbb{R}\}$ and let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. For $X \in \{M, \Sigma\}$ we will denote by $\mathcal{A}_X$ the space of all smooth $\mathfrak{g}$-valued 1-forms on X and by $\mathcal{G}_X$ the group of all smooth G-valued functions on X. In the special case $X = M$ we will often write $\mathcal{A}$ instead of $\mathcal{A}_X$ and $\mathcal{G}$ instead of $\mathcal{G}_X$.

We now fix a point $\sigma_0 \in \Sigma$ and a point $t_0 \in S^1$. In [23, 24] we consider only the special case $t_0 = 1 \in S^1$ ($\cong \{z \in \mathbb{C} \mid \|z\| = 1\}$). In the present paper we will not assume this anymore.

2.1. **Quasi-axial and torus gauge fixing: the basic idea.** In order to motivate the definition of quasi-axial gauge fixing for manifolds of the form $M = \Sigma \times S^1$ we first recall the definition of axial gauge fixing for manifolds of the form $M = \Sigma \times \mathbb{R}$.

Let $M = \Sigma \times \mathbb{R}$ and let $\frac{\partial}{\partial t}$ (resp. dt) denote the vector field (resp. 1-form) on $\mathbb{R}$ which is induced by $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$. By lifting $\frac{\partial}{\partial t}$ and dt to $M = \Sigma \times \mathbb{R}$ in the obvious way we obtain a vector field and a 1-form on M which will also be denoted by $\frac{\partial}{\partial t}$ resp. dt. Clearly, every $A \in \mathcal{A} = \mathcal{A}_M$ can be written uniquely in the form $A = A^\perp + A_0 dt$ with $A_0 \in C^\infty(M, \mathfrak{g})$ and $A^\perp \in \mathcal{A}^\perp := \{A \in \mathcal{A} \mid A(\frac{\partial}{\partial t}) = 0\}$.

Let us now consider manifolds M of the form $M = \Sigma \times S^1$. In this situation $\frac{\partial}{\partial t}$ will denote the vector field on S^1 which is induced by the curve

$$i_{S^1} : [0,1] \ni s \mapsto \exp(2\pi i s) \in \{z \in \mathbb{C} \mid \|z\| = 1\} \cong S^1 \tag{2.1}$$

and dt the 1-form on S^1 which is dual to $\frac{\partial}{\partial t}$. Again we can lift $\frac{\partial}{\partial t}$ and dt to a vector field resp. a 1-form on M, which will again be denoted by $\frac{\partial}{\partial t}$ resp. dt. As before every $A \in \mathcal{A}$ can be written uniquely in the form $A = A^\perp + A_0 dt$ with $A^\perp \in \mathcal{A}^\perp$ and $A_0 \in C^\infty(M, \mathfrak{g})$ where $\mathcal{A}^\perp$ is defined in total analogy to the $\Sigma \times \mathbb{R}$ case by

$$\mathcal{A}^\perp := \{A \in \mathcal{A} \mid A(\tfrac{\partial}{\partial t}) = 0\} \tag{2.2}$$

However, there is a crucial difference between the case $M = \Sigma \times \mathbb{R}$ and the case $M = \Sigma \times S^1$. For $M = \Sigma \times \mathbb{R}$ the condition $A_0 = 0$ (which is equivalent to the condition $A \in \mathcal{A}^\perp$) defines a gauge. More precisely: Every 1-form $A \in \mathcal{A}$ is gauge equivalent to a 1-form in $\mathcal{A}^\perp$. By contrast for $M = \Sigma \times S^1$ the condition $A_0 = 0$ does <u>not</u> define a gauge. There are 1-forms A which are not gauge equivalent to any 1-form in $\mathcal{A}^\perp$. For example this is the case for any 1-form A with the property that the holonomy $\mathcal{P} \exp(\int_{l_\sigma} A)$ is not equal to 1 for some $\sigma \in \Sigma$. Here l_σ denotes the "vertical" loop $[0,1] \ni s \mapsto (\sigma, i_{S^1}(s)) \in M$ "above" the fixed point $\sigma \in \Sigma$. This follows immediately from the two observations that, firstly, $\mathcal{P} \exp(\int_{l_\sigma} A^\perp) = 1$ for every $A^\perp \in \mathcal{A}^\perp$ and, secondly, for two gauge equivalent 1-forms $A_1 \in \mathcal{A}$ and $A_2 \in \mathcal{A}$ the corresponding holonomies $\mathcal{P} \exp(\int_{l_\sigma} A_1)$ and $\mathcal{P} \exp(\int_{l_\sigma} A_2)$ are conjugated to each other.

Thus, in order to obtain a proper gauge we have to weaken the condition $A_0 = 0$. There are two natural candidates for such a weakened condition.

1. Option: Instead of demanding $A_0(\sigma, t) = 0$ for all $\sigma \in \Sigma, t \in S^1$ we just demand that $A_0(\sigma, t)$ is independent of the second variable t, i.e. we demand that $A_0 = B$ holds where $B \in C^\infty(\Sigma, \mathfrak{g}) \subset C^\infty(M, \mathfrak{g})$ ("quasi-axial gauge fixing").

2. Option: We demand, firstly, that $A_0(\sigma, t)$ is independent of the variable t and, secondly, that it takes values in the Lie algebra $\mathfrak{t}$ of a fixed maximal torus $T \subset G$ ("torus gauge fixing").

Accordingly, let us introduce the spaces

$$\mathcal{A}^{qax} := \mathcal{A}^\perp \oplus \{B dt \mid B \in C^\infty(\Sigma, \mathfrak{g})\} \tag{2.3}$$

$$\mathcal{A}^{qax}(T) := \mathcal{A}^\perp \oplus \{B dt \mid B \in C^\infty(\Sigma, \mathfrak{t})\} \tag{2.4}$$

2.2. **Some technical details for quasi-axial gauge fixing.** Let us first analyze when/if quasi-axial gauge fixing really is a "proper" gauge fixing in the sense that every gauge field is gauge-equivalent to a "quasi-axial" gauge field. In order to answer this question we start with a fixed gauge field $A \in \mathcal{A}$ and try to find a $A^q = \mathcal{A}^\perp + B dt \in \mathcal{A}^{qax}$, $\mathcal{A}^\perp \in \mathcal{A}$, $B \in C^\infty(\Sigma, \mathfrak{g})$, and an element $\tilde{\Omega}$ of the subgroup[3] $\tilde{\mathcal{G}} := \{\Omega \in \mathcal{G} \mid \Omega(\sigma, 1) = 1 \text{ for all } \sigma \in \Sigma\}$ of $\mathcal{G}$ such that

$$A = A^q \cdot \tilde{\Omega} = (A^\perp + B dt) \cdot \tilde{\Omega} \tag{2.5}$$

---

[3]it is more convenient to work with the subgroup $\tilde{\mathcal{G}}$ of $\mathcal{G}$ here. We remark that $\mathcal{G}$ is the semi-direct product of $\tilde{\mathcal{G}}$ and $\mathcal{G}_\Sigma \subset \mathcal{G}$. The factor $\mathcal{G}_\Sigma$ will be taken care of in Sec. 2.4 below.

holds. Here "$\cdot$" denotes the standard right operation of $\mathcal{G}$ on $\mathcal{A}$ (the "gauge operation"). Taking into account that Eq. (2.5) implies

$$g_A(\sigma) := \mathcal{P}\exp(\int_{l_\sigma} A) = \mathcal{P}\exp(\int_{l_\sigma} A^\perp + Bdt) = \exp(B(\sigma)) \quad \forall \sigma \in \Sigma \tag{2.6}$$

where l_σ denotes again the "vertical" loop above the point σ it is clear that in order to find such a $A^q \in \mathcal{A}^{qax}$ one first has to find a lift $B : \Sigma \to \mathfrak{g}$ of $g_A : \Sigma \to G$ w.r.t. the projection $\exp : \mathfrak{g} \to G$. In order to find such a lift B it is tempting to apply the standard theory of coverings, see e.g. [29]. What complicates matters somewhat is that $\exp : \mathfrak{g} \to G$ is not a covering if G is Non-Abelian. On the other hand $\exp : S^* \to G_{reg}$ where G_{reg} denotes the set of all "regular"[4] elements of G and where S^* is any fixed connected component of $\exp^{-1}(G_{reg})$ is a (connected) covering. So if $g_A : \Sigma \to G$ takes only values in G_{reg} then we can apply the standard theory of coverings and conclude that at least in the following two situations there is a (smooth) lift $B : \Sigma \to S^*$ of g_A:

    i) Σ is simply-connected. In this case the existence of the lift B follows from the well-known "Lifting Theorem".

    ii) G is simply-connected. In this case the existence of the lift B follows from the fact that then also G_{reg} is simply-connected (cf. [15], Chap. V, Sec. 7) and, consequently, the covering $\exp : S^* \to G_{reg}$ is just a bijection.

Accordingly, let us assume for the rest of this paper that G or Σ is simply-connected.

Once such a lift B is found it is not difficult to find also a $\tilde{\Omega} \in \tilde{\mathcal{G}}$ and a $A^\perp$ such that (2.5) is fulfilled with $A^q := A^\perp + Bdt$. Thus if Σ or G is simply-connected then $\mathcal{A}_{reg} \subset \mathcal{A}^{qax} \cdot \tilde{\mathcal{G}}$ where

$$\mathcal{A}_{reg} := \{A \in \mathcal{A} \mid g_A : \Sigma \to G \text{ takes values in } G_{reg}\} \tag{2.7}$$

It can be shown that the codimension of the subset $G \backslash G_{reg}$ of G is at least 3. So in the special case when $\dim(\Sigma) = 2$ it is intuitively clear that for "almost all" $A \in \mathcal{A}$ the function g_A will take values in G_{reg}. In other words: the set $\mathcal{A} \backslash \mathcal{A}_{reg}$ is then "negligible". Accordingly, let us assume for the rest of this paper that Σ is 2-dimensional.

2.3. **The Faddeev-Popov determinant of quasi-axial gauge fixing.** The space $\mathcal{A}^{qax}$ can be characterized by

$$A \in \mathcal{A}^{qax} \quad \Leftrightarrow \quad F(A) = 0$$

where the function $F : \mathcal{A} \to C^\infty(M, \mathfrak{g})$ is given by $F(A) = \frac{\partial}{\partial t}A_0$. Taking into account that the set $\mathcal{A} \backslash \mathcal{A}_{reg}$ is "negligible" when $\dim(\Sigma) = 2$ we obtain, informally, for every gauge-invariant (i.e. $\mathcal{G}$-invariant) function $\chi : \mathcal{A} \to \mathbb{C}$

$$\int_{\mathcal{A}} \chi(A)DA = \int_{\mathcal{A}_{reg}} \chi(A)DA = \int_{\mathcal{A}_{reg}} \chi(A)\triangle_{FP}[A]\delta(F(A))DA \tag{2.8}$$

where DA is the informal "Lebesgue measure" on $\mathcal{A}$ and $\triangle_{FP}$ the Faddeev-Popov-determinant associated to F, which is given explicitly by $\triangle_{FP}[A] := \det\left(\frac{\delta F(A \cdot \Omega)}{\delta \Omega}\big|_{\Omega = \Omega_A}\right)$ for all $A \in \mathcal{A}$ with $\Omega_A \in \mathcal{G}$ given by $F(A \cdot \Omega_A) = 0$.

As the informal measure "$\delta(F(A))DA$" is concentrated on $\{A \in \mathcal{A} \mid F(A) = 0\} = \mathcal{A}^{qax}$ we need to know $\triangle_{FP}[A]$ only in the special case $A \in \mathcal{A}^{qax}$, i.e. for A of the form $A = A^\perp + Bdt$, $A^\perp \in \mathcal{A}^\perp$, $B \in C^\infty(\Sigma, \mathfrak{g})$. Clearly, for such a A we have $\Omega_A = 1$. A short computation shows that

$$\triangle_{FP}(A^\perp + Bdt)$$
$$= \det\left(\frac{\delta F((A^\perp + Bdt)\cdot \Omega)}{\delta \Omega}\big|_{\Omega = 1}\right) = \left|\det\left(\left(\frac{\partial}{\partial t} + \mathrm{ad}(B)\right)\cdot\frac{\partial}{\partial t}\right)\right| \sim \left|\det\left(\frac{\partial}{\partial t} + \mathrm{ad}(B)\right)\right| =: \triangle[B]$$

It is natural to assume that the informal measure $\delta(F(A))DA$ on $\mathcal{A}^{qax} = \mathcal{A}^\perp \oplus C^\infty(\Sigma, \mathfrak{g})dt$ is of the form

$$\delta(F(A))DA = DA^\perp \otimes f(B)DB$$

----
[4] i.e. the set of all $g \in G$ such that g is contained in a unique maximal torus of G

where $DA^\perp$ resp. DB is the informal Lebesgue measure on $\mathcal{A}^\perp$ resp. $C^\infty(\Sigma, \mathfrak{g})$ and f a suitable function on $C^\infty(\Sigma, \mathfrak{g})$. Making this assumption we obtain

$$\int_\mathcal{A} \chi(A)DA \sim \int_{C^\infty(\Sigma,\mathfrak{g})} \left[\int_{\mathcal{A}^\perp} \chi(A^\perp + Bdt)DA^\perp\right] \triangle[B]f(B)DB \qquad (2.9)$$

Another rather natural assumption is that the image of $f(B)DB$ under the mapping $C^\infty(\Sigma, \mathfrak{g}) \ni B \mapsto \exp(B) \in C^\infty(\Sigma, G)$ should coincide with the informal Haar measure Dg on $C^\infty(\Sigma, G)$ (outside the set $C^\infty(\Sigma, G\backslash G_{reg})$ which we consider "negligible"). Taking into account that the differential $d\exp(x)$ of $\exp : \mathfrak{g} \to G$ in a point $x \in \mathfrak{g}$ is given by[5]

$$d\exp(x) = \exp(x) \cdot \sum_{n=0}^\infty \frac{(\mathrm{ad}(x))^n}{(n+1)!}$$

one can then conclude at an informal level that $f(B) = \det\left(\sum_{n=0}^\infty \frac{(\mathrm{ad}(B))^n}{(n+1)!}\right)$.

A closer look at the heuristic arguments used so far shows that the integration $\int_{C^\infty(\Sigma,\mathfrak{g})} \cdots$ in Eq. (2.9) above can be replaced by $\int_{C^\infty(\Sigma,S^*)} \cdots$ for an arbitrary connected component S^* of $\exp^{-1}(G_{reg})$ (cf. Sec. 4 in [23]). Doing this we finally obtain

$$\int_\mathcal{A} \chi(A)DA \sim \int_{C^\infty(\Sigma,S^*)} \left[\int_{\mathcal{A}^\perp} \chi(A^\perp + Bdt)DA^\perp\right]$$
$$\times \left|\det(\partial/\partial t + \mathrm{ad}(B))\right| \det\left(\sum_{n=0}^\infty \frac{(\mathrm{ad}(B))^n}{(n+1)!}\right)DB \quad (2.10)$$

2.4. **From quasi-axial to torus gauge fixing.** Let us fix an Ad-invariant scalar product $(\cdot,\cdot)_\mathfrak{g}$ on $\mathfrak{g}$ and, once and for all, a maximal torus T of G. The Lie algebra of T will be denoted by $\mathfrak{t}$. We set

$$\mathfrak{g}_0 := \mathfrak{t}^\perp \qquad (2.11)$$

where $\mathfrak{t}^\perp$ denotes the $(\cdot,\cdot)_\mathfrak{g}$-orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$. Moreover, let us fix an open "alcove" (or "affine Weyl chamber") $P \subset \mathfrak{t}$ and set $S^* := P \cdot G$ where "$\cdot$" denotes the right operation of G on $\mathfrak{g}$ given by $B \cdot g = g^{-1}Bg$. (Note that $P \cdot G$ is indeed a connected component of $\exp^{-1}(G_{reg})$). We then have

$$P \cong S^*/G \qquad (2.12)$$
$$\pi_*(dx) = \det(-\mathrm{ad}(x)_{|\mathfrak{g}_0})dx \qquad (2.13)$$

Here $\pi : S^* \to S^*/G \cong P$ is the canonical projection, dx denotes both the restriction of Lebesgue measure on $\mathfrak{t}$ onto P and the restriction of Lebesgue measure on $\mathfrak{g}$ onto S^*, and, finally, $\pi_*(dx)$ is the image of the measure dx on S^* under the projection π. In view of $\mathcal{G}_\Sigma = C^\infty(\Sigma, G)$ and Eqs. (2.12) and (2.13) one could expect naively that

$$C^\infty(\Sigma, P) \cong C^\infty(\Sigma, S^*)/\mathcal{G}_\Sigma \qquad (2.14)$$
$$\pi_*(DB) = \det(-\mathrm{ad}(B)_{|\mathfrak{g}_0})DB \qquad (2.15)$$

holds where $\pi : C^\infty(\Sigma, S^*) \to C^\infty(\Sigma, S^*)/\mathcal{G}_\Sigma$ is the canonical projection and where DB denotes both the restriction of the informal "Lebesgue measure" on $C^\infty(\Sigma, \mathfrak{g})$ onto $C^\infty(\Sigma, S^*)$ and the restriction of the informal "Lebesgue measure" on $C^\infty(\Sigma, \mathfrak{t})$ onto $C^\infty(\Sigma, P)$.

However, there are well-known topological obstructions (cf. [12], [23]), which prevent Eq. (2.14) from holding in general. Before we take a closer look at these obstructions in the general case let us restrict ourselves for a while to those (special) situations where Eq. (2.14) does hold (this will be the case if Σ is non-compact, cf. the discussion below). As the operation of $\mathcal{G}_\Sigma$ on $\mathcal{A}$ is linear and as it leaves the subspace $\mathcal{A}^\perp$ of $\mathcal{A}$ and the informal measure $DA^\perp$ on $\mathcal{A}^\perp$ invariant we can conclude, informally, that the function $\tilde{\chi}(B) : C^\infty(\Sigma, \mathfrak{g}) \ni B \mapsto \int \chi(A^\perp + Bdt)DA^\perp \in \mathbb{C}$ is $\mathcal{G}_\Sigma$-invariant (here χ is as in Subsec. 2.3).

---

[5]in this paper Ad denotes the representation $G \ni g \mapsto g^{-1}(\cdot)g \in \mathrm{End}(\mathfrak{g})$ and ad the derived representation $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. Thus ad is not given by the usual formula $\mathrm{ad}(A)(B) = [A, B]$, $A, B \in \mathfrak{g}$, but by the formula $\mathrm{ad}(A)(B) = [B, A]$

Moreover, on can argue that the functions $\triangle[B]$ and $\det\left(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}(B))^n}{(n+1)!}\right)$ on $C^\infty(\Sigma, \mathfrak{g})$ are $\mathcal{G}_\Sigma$-invariant, too. In the special situations where Eq. (2.14) holds we therefore obtain, informally,

$$
\int_{\mathcal{A}} \chi(A) DA \sim \int_{C^\infty(\Sigma, S^*)} \tilde{\chi}(B) \triangle[B] \det\left(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}(B))^n}{(n+1)!}\right) DB
$$

$$
\overset{(*)}{=} \int_{C^\infty(\Sigma, P)} \tilde{\chi}(B) \triangle[B] \det\left(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}(B))^n}{(n+1)!}\right) \cdot \det\left(-\mathrm{ad}(B)_{|\mathfrak{g}_0}\right) DB
$$

$$
\overset{(**)}{=} \int_{C^\infty(\Sigma, P)} \int \chi(A^\perp + Bdt) DA^\perp \det\left(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\right) \triangle[B] DB \quad (2.16)
$$

Here step $(*)$ follows from Eqs. (2.14), (2.15) and the relations $\tilde{\chi} = \tilde{\chi} \circ \pi$, $\triangle[\cdot] = \triangle[\cdot] \circ \pi$, and $\det(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}(\cdot))^n}{(n+1)!}) = \det(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}(\cdot))^n}{(n+1)!}) \circ \pi$. Step $(**)$ follows because $\det(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}(B))^n}{(n+1)!}) = \det(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}(B)_{|\mathfrak{g}_0})^n}{(n+1)!}))$.

Let us now go back to the general case where, because of the topological obstructions mentioned above, Eq. (2.14) need not hold. In order to find a suitable generalization of Eq. (2.16) we now consider the bijection $P \times G/T \ni (B, gT) \mapsto gBg^{-1} \in S^*$. Clearly, this bijection induces a bijection $C^\infty(\Sigma, P) \times C^\infty(\Sigma, G/T) \to C^\infty(\Sigma, S^*)$, so we can identify the space $C^\infty(\Sigma, P) \times C^\infty(\Sigma, G/T)$ with the space $C^\infty(\Sigma, S^*)$. After identifying these two spaces the operation of $\mathcal{G}_\Sigma$ on $C^\infty(\Sigma, P) \times C^\infty(\Sigma, G/T) \cong C^\infty(\Sigma, S^*)$ can be written in the form $(B, \bar{g}) \cdot \Omega = (B, \Omega^{-1}\bar{g})$ from which

$$
C^\infty(\Sigma, S^*)/\mathcal{G}_\Sigma \cong C^\infty(\Sigma, P) \times (C^\infty(\Sigma, G/T)/\mathcal{G}_\Sigma) \quad (2.17)
$$

follows. For a proof of the following proposition see [24].

Proposition 2.1. *Under the assumptions made in Subsec. 2.2, i.e. $\dim(\Sigma) = 2$ and G or Σ is simply-connected, we have*

$$
C^\infty(\Sigma, G/T)/\mathcal{G}_\Sigma = [\Sigma, G/T]
$$

Let us now fix for the rest of this paper a representative $\bar{g}_\mathrm{h} \in C^\infty(\Sigma, G/T)$ for each homotopy class $\mathrm{h} \in [\Sigma, G/T]$. For $\bar{g} = gT \in G/T$ we will denote by $\bar{g}B\bar{g}^{-1}$ the element gBg^{-1} of G (which clearly does not depend on the special choice of g). Taking into account that

$$
\triangle(B) \det\left(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\right) = \triangle((\bar{g}_\mathrm{h} \cdot B \cdot \bar{g}_\mathrm{h}^{-1})) \det\left(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}((\bar{g}_\mathrm{h} \cdot B \cdot \bar{g}_\mathrm{h}^{-1}))_{|\mathfrak{g}_0}))\right)
$$

one can derive (at a heuristic level) the following generalization of Eq. (2.16) above (for more details see [24])

$$
\int_{\mathcal{A}} \chi(A) DA \sim \sum_{\mathrm{h} \in [\Sigma, G/T]} \int_{C^\infty(\Sigma, P)} \left[\int_{\mathcal{A}^\perp} \chi(A^\perp + (\bar{g}_\mathrm{h} \cdot B \cdot \bar{g}_\mathrm{h}^{-1})dt) DA^\perp\right]
$$

$$
\times \triangle(B) \det\left(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\right) DB \quad (2.18)
$$

Note that because of $C^\infty(\Sigma, G/T)/\mathcal{G}_\Sigma = [\Sigma, G/T]$ and the $\mathcal{G}_\Sigma$-invariance of $\tilde{\chi}(B) = \int_{\mathcal{A}^\perp} \chi(A^\perp + Bdt) DA^\perp$ the expression $\int_{\mathcal{A}^\perp} \chi(A^\perp + (\bar{g}_\mathrm{h} \cdot B \cdot \bar{g}_\mathrm{h}^{-1})dt) DA^\perp$ above does not depend on the special choice of $\bar{g}_\mathrm{h}$.

If Σ is non-compact then all continuous mappings $\Sigma \to G/T$ are homotopic to each other. In other words, we then have $[\Sigma, G/T] = \{[1_T]\}$ where $1_T : \Sigma \to G/T$ is the constant map taking only the value $T \in G/T = \{gT \mid g \in G\}$. So in this special situation Eq. (2.18) reduces to Eq. (2.16). For compact Σ, however, we will have to work with Eq. (2.18). Thus for compact Σ, the functions $\bar{g}_\mathrm{h} \cdot B \cdot \bar{g}_\mathrm{h}^{-1}$ will in general not take only values in $\mathfrak{t}$. This reduces the usefulness of Eq. (2.18) considerably. Fortunately, for many functions χ it is possible to derive an "Abelian version" of Eq. (2.18), as we will now show.

2.5. **A useful modification of Eq.** (2.18) **for compact** Σ. Recall that we have fixed a point $\sigma_0 \in \Sigma$. Clearly, the restriction mapping $\mathcal{G}_\Sigma \ni \Omega \mapsto \Omega_{|\Sigma \setminus \{\sigma_0\}} \in \mathcal{G}_{\Sigma \setminus \{\sigma_0\}}$ is injective so we can identify $\mathcal{G}_\Sigma$ with a subgroup of $\mathcal{G}_{\Sigma \setminus \{\sigma_0\}}$. Similarly, let us identify the spaces $\mathcal{A}^\perp$, $\mathcal{A}^{qax}$, $C^\infty(\Sigma, G/T)$, and $C^\infty(\Sigma, \mathfrak{g})$ with the obvious subspaces of $\mathcal{A}^\perp_{(\Sigma \setminus \{\sigma_0\}) \times S^1}$ resp. $\mathcal{A}^{qax}_{(\Sigma \setminus \{\sigma_0\}) \times S^1}$ resp. $C^\infty(\Sigma \setminus \{\sigma_0\}, G/T)$ resp. $C^\infty(\Sigma \setminus \{\sigma_0\}, \mathfrak{g})$.

As $\Sigma \setminus \{\sigma_0\}$ is noncompact every $\bar{g} \in C^\infty(\Sigma \setminus \{\sigma_0\}, G/T)$ is 0-homotopic and can therefore be lifted to an element of $C^\infty(\Sigma \setminus \{\sigma_0\}, G) = \mathcal{G}_{\Sigma \setminus \{\sigma_0\}}$, i.e. there is always a $\Omega \in \mathcal{G}_{\Sigma \setminus \{\sigma_0\}}$ such that $\bar{g} = \pi_{G/T} \circ \Omega$ where $\pi_{G/T} : G \to G/T$ is the canonical projection. We will now pick for each $h \in [\Sigma, G/T]$ such a lift $\Omega_h \in \mathcal{G}_{\Sigma \setminus \{\sigma_0\}}$ of the representative $\bar{g}_h \in C^\infty(\Sigma, G/T) \subset C^\infty(\Sigma \setminus \{\sigma_0\}, G/T)$ of h which we have fixed above.

Let $\chi : \mathcal{A} \to \mathbb{C}$ be a $\mathcal{G}$-invariant function. The space $\mathcal{A}^{qax} \subset \mathcal{A}$ is clearly $\mathcal{G}_\Sigma$-invariant so $\mathcal{G}_\Sigma$ operates on $\mathcal{A}^{qax}$ and the function $\chi^{qax} := \chi_{|\mathcal{A}^{qax}}$ in invariant under this operation. Let us now make the additional assumption that $\chi^{qax} : \mathcal{A}^{qax} \to \mathbb{C}$ can be extended to a function $\overline{\chi^{qax}} : \mathcal{A}^{qax}_{(\Sigma \setminus \{\sigma_0\}) \times S^1} \to \mathbb{C}$ which is $\mathcal{G}_{\Sigma \setminus \{\sigma_0\}}$-invariant, or at least $\overline{\mathcal{G}_\Sigma}$-invariant, where $\overline{\mathcal{G}_\Sigma}$ is the subgroup of $\mathcal{G}_{\Sigma \setminus \{\sigma_0\}}$ which is generated by $\mathcal{G}_\Sigma$ and all Ω_h, $h \in [\Sigma, G/T]$. Then we obtain for the integrand in the inner integral on the right-hand side of (2.18)

$$\chi(A^\perp + (\bar{g}_h \cdot B \cdot \bar{g}_h^{-1})dt) = \chi^{qax}(A^\perp + (\bar{g}_h \cdot B \cdot \bar{g}_h^{-1})dt) = \overline{\chi^{qax}}(A^\perp + (\Omega_h \cdot B \cdot \Omega_h^{-1})dt)$$
$$= \overline{\chi^{qax}}((A^\perp \cdot \Omega_h) + B \cdot dt) = \overline{\chi^{qax}}(\Omega_h^{-1}A^\perp \Omega_h + \Omega_h^{-1}d\Omega_h + B \cdot dt) \quad (2.19)$$

Thus, for such a function χ we arrive at the following useful modification of (2.18)

$$\int_\mathcal{A} \chi(A)DA \sim \sum_{h \in [\Sigma, G/T]} \int_{C^\infty(\Sigma, P)} \left[\int_{\mathcal{A}^\perp} \overline{\chi^{qax}}(\Omega_h^{-1}A^\perp \Omega_h + \Omega_h^{-1}d\Omega_h + B \cdot dt)DA^\perp \right]$$
$$\times \triangle(B) \det\big(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\big)DB \quad (2.20)$$

2.6. **Identification of** $[\Sigma, G/T]$ **for compact oriented surfaces** Σ. Recall that we have been assuming that $\dim(\Sigma) = 2$. Let us now assume additionally that Σ is oriented and compact. Moreover, let us assume for simplicity that G is simply-connected (the case where not G is assumed to be simply-connected but Σ is covered by Remark 2.1 below). Then there is a natural bijection from the set $[\Sigma, G/T]$ onto $\pi_2(G/T) \cong \ker(\exp_{|\mathfrak{t}}) \cong \mathbb{Z}^r$ where $r = \mathrm{rank}(G)$, see [12] and [15], Chap. V, Sec. 7. Instead of recalling the abstract definition of this bijection $[\Sigma, G/T] \to \ker(\exp_{|\mathfrak{t}})$ we will give a more concrete description, which will be more useful for our purposes.

Let, for any fixed auxiliary Riemannian metric on Σ, $B_\epsilon(\sigma_0)$ denote the closed ball around σ_0 with radius ϵ. It is not difficult to see that for each $h \in [\Sigma, G/T]$ the limit

$$\int_{\Sigma \setminus \sigma_0} d(\Omega_h^{-1}d\Omega_h) := \lim_{\epsilon \to 0} \int_{\Sigma \setminus B_\epsilon(\sigma_0)} d(\Omega_h^{-1}d\Omega_h) \quad (2.21)$$

exists and is independent of the choice of the auxiliary Riemannian metric. Let us set

$$n(\Omega_h) := \pi_\mathfrak{t}\big(\int_{\Sigma \setminus \{\sigma_0\}} d(\Omega_h^{-1}d\Omega_h)\big) \in \mathfrak{t} \quad (2.22)$$

where $\pi_\mathfrak{t}$ denotes the orthogonal projection $\mathfrak{g} \to \mathfrak{t}$. Then we have

Proposition 2.2.   i) $n(\Omega_h)$ *depends neither on the special choice of the lift* Ω_h *of* $(\bar{g}_h)_{|\Sigma \setminus \{\sigma_0\}}$
      nor on the special choice of the representative $\bar{g}_h \in C^\infty(\Sigma, G/T)$ *of* h. *It only depends*
      on h. *Thus we can set* $n(h) := n(\Omega_h)$.
    ii) *If* G *is simply-connected then the mapping* $[\Sigma, G/T] \ni h \mapsto n(h) \in \mathfrak{t}$ *is a bijection from*
      $[\Sigma, G/T]$ *onto* $\ker(\exp_{|\mathfrak{t}})$. *In particular, we then have*

$$\{n(h) \mid h \in [\Sigma, G/T]\} = \{B \in \mathfrak{t} \mid \exp(B) = 1\} \quad (2.23)$$

For an elementary proof of this proposition, see [24] (cf. also Sec. 5 in [12] for a very similar result).

Remark 2.1. The first part of Proposition 2.2 holds also when G is not assumed to be simply-connected. However, in this case the second part must modified. More precisely, if G is not simply-connected then the mapping $[\Sigma, G/T] \ni \mathrm{h} \mapsto n(\mathrm{h}) \in \mathfrak{t}$ is a bijection from $[\Sigma, G/T]$ onto the subgroup Γ of $\ker(\exp_{|\mathfrak{t}})$ which is generated by the inverse roots, cf. [15], Chap. V, Sec. 7.

3. Torus gauge fixing applied to Chern-Simons models on $\Sigma \times S^1$

3.1. **Chern-Simons models and Wilson loop observables.**
For the rest of this paper we will not only assume that $\dim(\Sigma) = 2$ but, additionally, that Σ is compact and oriented. Moreover, G will be assumed to be either Abelian or simply-connected and simple (in the former case we will also assume that $\Sigma \cong S^2$).

Without loss of generality we can assume that G is a Lie subgroup of $U(N)$, $N \in \mathbb{N}$. The Lie algebra $\mathfrak{g}$ of G can then be identified with the obvious Lie subalgebra of the Lie algebra $u(N)$ of $U(N)$.

For the rest of this paper we will fix an integer $k \in \mathbb{Z}\backslash\{0\}$ and set

$$\lambda := \tfrac{1}{k}.$$

With the assumptions above $M = \Sigma \times S^1$ is an oriented compact 3-manifold. Thus the Chern-Simons action function S_{CS} corresponding to the triple (M, G, k) is well-defined and given by

$$S_{CS}(A) = \tfrac{k}{4\pi} \int_M \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A), \quad A \in \mathcal{A}$$

with $\mathrm{Tr} := c \cdot \mathrm{Tr}_{\mathrm{Mat}(N,\mathbb{C})}$ where c is a suitable normalization constant chosen such that the exponential $\exp(iS_{CS})$ is $\mathcal{G}$-invariant. If G is Abelian or if $G = SU(N)$ (which are the only two types of groups for which we will make concrete computations below) this will be the case, e.g. if $c = 1$, which is the value of c that we will choose for these groups.

From the definition of S_{CS} it is obvious that S_{CS} is invariant under (orientation-preserving) diffeomorphisms. Thus, at a heuristic level, we can expect that the heuristic integral (the "partition function")

$$Z(M) := \int \exp(iS_{CS}(A))DA$$

is a topological invariant of the 3-manifold M. Here DA denotes again the informal "Lebesgue measure" on the space $\mathcal{A}$.

Similarly, we can expect that the mapping which maps every sufficiently "regular" colored link $L = ((l_1, l_2, \ldots, l_n), (\rho_1, \rho_2, \ldots, \rho_n))$ in M to the heuristic integral (the "Wilson loop observable" associated to L)

$$\mathrm{WLO}(L) := \frac{1}{Z(M)} \int \prod_i \mathrm{Tr}_{\rho_i}\big(\mathcal{P} \exp\big(\int_{l_i} A\big)\big) \exp(iS_{CS}(A))DA \tag{3.1}$$

is a link invariant (or, rather, an invariant of colored links). Here $\mathcal{P} \exp\big(\int_{l_i} A\big)$ denotes the holonomy of A around the loop l_i and Tr_{ρ_i}, $i \leq n$, the trace in the finite-dimensional representation ρ_i of G.

For the rest of this paper, we will now fix a "sufficiently regular" colored link

$$L = ((l_1, l_2, \ldots, l_n), (\rho_1, \rho_2, \ldots, \rho_n))$$

in M and set $\underline{\rho} := (\rho_1, \rho_2, \ldots, \rho_n)$. The "uncolored" link $(l_1, l_2, \ldots, l_n)$ will also be denoted by L. In order to make precise what we mean with "sufficiently regular" above we will introduce the following definitions:

Let π_Σ (resp. π_{S^1}) denote the canonical projection $\Sigma \times S^1 \to \Sigma$ (resp. $\Sigma \times S^1 \to S^1$). For each $j \leq n$ we will set $l_\Sigma^j := \pi_\Sigma \circ l_j$ and $l_{S^1}^j := \pi_{S^1} \circ l_j$. Similarly, we will set $c_\Sigma := \pi_\Sigma \circ c$ and $c_{S^1} := \pi_{S^1} \circ c$ for an arbitrary curve c in $\Sigma \times S^1$. We will call $p \in \Sigma$ a "double point" (resp. a "triple point") of L if the intersection of $\pi_\Sigma^{-1}(\{p\})$ with the union of the arcs of $l_1, l_2, \ldots l_n$ contains at least two (resp. at least three) elements. The set of double points of L will be denoted by $DP(L)$. We will assume in the sequel (with the exception of Subsec. 6.2 below where we study "vertical" links) that the link L is "admissible" in the following sense:

(A1) There are only finitely many double points and no triple points of L

(A2) For each $p \in DP(L)$ the corresponding tangent vectors, i.e. the vectors $(l_\Sigma^i)'(\bar{t})$ and $(l_\Sigma^j)'(\bar{u})$ in $T_p\Sigma$ where $\bar{t}, \bar{u} \in [0,1]$, $i,j \leq n$, are given by $p = l_\Sigma^i(\bar{t}) = l_\Sigma^j(\bar{u})$, are not parallel to each other.

(A3) For each $j \leq n$ the set $I_j(t_0) := (l_{S^1}^j)^{-1}(\{t_0\})$ is finite.

(A4) There is no $x \in \bigcup_j \mathrm{arc}(l_j)$ such that simultaneously $\pi_{S^1}(x) = t_0$ and $\pi_\Sigma(x) \in DP(L)$ holds.

Note that from (A1) it follows that the set $\Sigma \backslash (\bigcup_j \mathrm{arc}(l_\Sigma^j))$ has only finitely many connected components. We will denote these connected components by $X_1, X_2, \ldots, X_\mu$, $\mu \in \mathbb{N}$, in the sequel.

3.2. **The identification** $\mathcal{A}^\perp \equiv C^\infty(S^1, \mathcal{A}_\Sigma)$ **and the Hilbert spaces** $\mathcal{H}_\Sigma$, $\mathcal{H}^\perp$. Before we apply the results of Sec. 2 to the Chern-Simons action function it is useful to introduce some additional spaces. For every real vector space V let $\mathcal{A}_{\Sigma,V}$ denote the space of smooth V-valued 1-forms on Σ. We set $\mathcal{A}_\Sigma := \mathcal{A}_{\Sigma,\mathfrak{g}}$. We will call a function $\alpha : S^1 \to \mathcal{A}_{\Sigma,V}$ "smooth" if for every C^∞-vector field X on Σ the function $\Sigma \times S^1 \ni (\sigma, t) \mapsto \alpha(t)(X_\sigma) \in V$ is C^∞ and we will set $C^\infty(S^1, \mathcal{A}_{\Sigma,V}) := \{\alpha \mid \alpha : S^1 \to \mathcal{A}_{\Sigma,V} \text{ is smooth}\}$. $\frac{\partial}{\partial t}$ will denote the obvious operator on $C^\infty(S^1, \mathcal{A}_{\Sigma,V})$. During the rest of this paper we will identify $\mathcal{A}^\perp$ with $C^\infty(S^1, \mathcal{A}_\Sigma)$ in the obvious way. In particular, if $A^\perp \in \mathcal{A}^\perp$ and $t \in S^1$ then $A^\perp(t)$ will denote an element of $\mathcal{A}_\Sigma$.

In the sequel we will assume that the Ad-invariant scalar product $(\cdot, \cdot)_\mathfrak{g}$ on $\mathfrak{g}$ fixed above is the one given by

$$(A, B)_\mathfrak{g} = -\mathrm{Tr}(AB) \quad \text{for all } A, B \in \mathfrak{g} \tag{3.2}$$

Moreover, we fix a $(\cdot, \cdot)_\mathfrak{g}$-orthonormal basis $(T_a)_{a \leq \dim(G)}$ with the property that $T_a \in \mathfrak{t}$ for all $a \leq r := \mathrm{rank}(G) = \dim(T)$. $((T_a)_{a \leq \dim(G)}$ will be relevant in the concrete computations in Secs 5 and 6).

Let us also fix an auxiliary Riemannian metric $\mathbf{g}$ on Σ for the rest of this paper. $\mu_\mathbf{g}$ will denote the Riemannian volume measure on Σ associated to $\mathbf{g}$, $(\cdot, \cdot)_{\mathbf{g},\mathfrak{g}}$ the fibre metric on the bundle $\mathrm{Hom}(T(\Sigma), \mathfrak{g}) \cong T^*(\Sigma) \otimes \mathfrak{g}$ induced by $\mathbf{g}$ and $(\cdot, \cdot)_\mathfrak{g}$, and $\mathcal{H}_\Sigma$ the Hilbert space $\mathcal{H}_\Sigma := L^2\text{-}\Gamma(\mathrm{Hom}(T(\Sigma), \mathfrak{g}), \mu_\mathbf{g})$ of L^2-sections of the bundle $\mathrm{Hom}(T(\Sigma), \mathfrak{g})$ w.r.t. the measure $\mu_\mathbf{g}$ and the fibre metric $(\cdot, \cdot)_{\mathbf{g},\mathfrak{g}}$. The scalar product $\ll \cdot, \cdot \gg_{\mathcal{H}_\Sigma}$ of $\mathcal{H}_\Sigma$ is, of course, given by

$$\ll \alpha_1, \alpha_2 \gg_{\mathcal{H}_\Sigma} = \int_\Sigma (\alpha_1, \alpha_2)_{\mathbf{g},\mathfrak{g}} d\mu_\mathbf{g} \quad \forall \alpha_1, \alpha_2 \in \mathcal{H}_\Sigma$$

Finally, we set $\mathcal{H}^\perp := L^2_{\mathcal{H}_\Sigma}(S^1, dt)$, i.e. $\mathcal{H}^\perp$ is the space of $\mathcal{H}_\Sigma$-valued (measurable) functions on S^1 which are square-integrable w.r.t. dt. The scalar product $\ll \cdot, \cdot \gg_{\mathcal{H}^\perp}$ on $\mathcal{H}^\perp$ is given by

$$\ll A_1^\perp, A_2^\perp \gg_{\mathcal{H}^\perp} = \int_{S^1} \ll A_1^\perp(t), A_2^\perp(t) \gg_{\mathcal{H}_\Sigma} dt \quad \text{for all } A_1^\perp, A_2^\perp \in \mathcal{H}^\perp$$

By $\star$ we will denote four different operators: firstly, the Hodge star operator $\star : \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma$, secondly the operator $\star : C^\infty(S^1, \mathcal{A}_\Sigma) \to C^\infty(S^1, \mathcal{A}_\Sigma)$ defined by $(\star A^\perp)(t) = \star(A^\perp(t))$ for all $t \in S^1$, thirdly the operator $\mathcal{H}^\perp \to \mathcal{H}^\perp$ obtained by continuously extending $\star : C^\infty(S^1, \mathcal{A}_\Sigma) \to C^\infty(S^1, \mathcal{A}_\Sigma)$ to all of $\mathcal{H}^\perp$ and, finally, the Hodge operator $\Omega^2(\Sigma, \mathfrak{g}) \to C^\infty(\Sigma, \mathfrak{g})$ where $\Omega^2(\Sigma, \mathfrak{g})$ denotes the space of $\mathfrak{g}$-valued 2-forms on Σ.

The four analogous mappings obtained by replacing the surface Σ by $\Sigma \backslash \{\sigma_0\}$ will also be denoted by $\star$.

3.3. **Application of formula** (2.20). The restriction of the Chern-Simons action function S_{CS} onto the space $\mathcal{A}^{qax}$ is rather simple. More precisely, we have

Proposition 3.1. *Let* $A^\perp \in \mathcal{A}^\perp$ *and* $B \in C^\infty(\Sigma, \mathfrak{g})$. *Then*

$$S_{CS}(A^\perp + Bdt) = -\frac{k}{4\pi}\left[\ll A^\perp, \star(\tfrac{\partial}{\partial t} + \mathrm{ad}(B))A^\perp \gg_{\mathcal{H}^\perp} -2 \ll A^\perp, \star dB \gg_{\mathcal{H}^\perp}\right] \tag{3.3}$$

Proof. It is not difficult to see that for all $A^\perp \in \mathcal{A}^\perp$ and $A^{||} \in \{A_0 dt \mid A_0 \in C^\infty(M, \mathfrak{g})\}$ one has $S_{CS}(A^\perp + A^{||}) = \frac{k}{4\pi}\left[\int_M \text{Tr}(A^\perp \wedge dA^\perp) + 2\int_M \text{Tr}(A^\perp \wedge A^{||} \wedge A^\perp) + 2\int_M \text{Tr}(A^\perp \wedge dA^{||})\right]$. By applying this formula to the special case where $A^{||} = Bdt$ and taking into account the definitions of $\star$ and $\ll \cdot, \cdot \gg_{\mathcal{H}^\perp}$ the assertion follows (cf. Prop. 5.2 in [23]). $\qquad\square$

From Eq. (2.18) we obtain

$$\text{WLO}(L) = \frac{1}{Z(M)} \int \prod_i \text{Tr}_{\rho_i}\left(\mathcal{P}\exp\left(\int_{l_i} A\right)\right) \exp(iS_{CS}(A)) DA$$

$$\sim \sum_{\text{h}\in[\Sigma, G/T]} \int_{C^\infty(\Sigma, P)} \left[\int_{\mathcal{A}^\perp} \prod_i \text{Tr}_{\rho_i}\left(\mathcal{P}\exp\left(\int_{l_i} A^\perp + (\bar{g}_\text{h} B \bar{g}_\text{h}^{-1})dt\right)\right)\right]$$

$$\times \exp(iS_{CS}(A^\perp + (\bar{g}_\text{h} B \bar{g}_\text{h}^{-1})dt)) DA^\perp \right] \triangle[B] \det\left(1_{\mathfrak{g}_0} - \exp(\text{ad}(B)_{|\mathfrak{g}_0})\right) DB$$

We would now like to apply formula (2.20) above and obtain an "Abelian version" of the equation above. Before we can do this we have to extend the two $\mathcal{G}_\Sigma$-invariant functions

$$\mathcal{A}^{qax} \ni A^q \mapsto \prod_i \text{Tr}_{\rho_i}\left(\mathcal{P}\exp\left(\int_{l_i} A^q\right)\right) \in \mathbb{C} \tag{3.4}$$

$$\mathcal{A}^{qax} \ni A^q \mapsto S_{CS}(A^q) \in \mathbb{C} \tag{3.5}$$

to $\overline{\mathcal{G}_\Sigma}$-invariant functions on $\mathcal{A}^{qax}_{(\Sigma\backslash\{\sigma_0\})\times S^1}$.

If σ_0 is not in the image of the loops l_Σ^j, which we will assume in the sequel, then the expression on the right-hand side of Eq. (3.4) makes sense for arbitrary $A^q \in \mathcal{A}^{qax}_{(\Sigma\backslash\{\sigma_0\})\times S^1}$ and thus defines a $\mathcal{G}_{\Sigma\backslash\{\sigma_0\}}$-invariant function on $\mathcal{A}^{qax}_{(\Sigma\backslash\{\sigma_0\})\times S^1}$. The second function is just the restriction $(S_{CS})_{|\mathcal{A}^{qax}}$. Let $\overline{S}_{CS} : \mathcal{A}^{qax}_{|(\Sigma\backslash\{\sigma_0\})\times S^1} \to \mathbb{C}$ be given by

$$\overline{S}_{CS}(A^\perp + Bdt)$$
$$:= -\lim_{\epsilon \to 0} \frac{k}{4\pi} \int_{S^1} \int_{\Sigma\backslash B_{\epsilon(\sigma_0)}} \left[\left(A^\perp(t), \star\left(\frac{\partial}{\partial t} + \text{ad}(B)\right)A^\perp(t)\right)_{\mathfrak{g},\mathfrak{g}} - 2\text{Tr}(\star dA^\perp(t)B)\right] d\mu_\mathbf{g} dt,$$

for all $A^\perp \in \mathcal{A}^\perp_{(\Sigma\backslash\{\sigma_0\})\times S^1}$ and $B \in C^\infty(\Sigma\backslash\{\sigma_0\}, \mathfrak{g})$ for which the limit $\epsilon \to 0$ exists[6], and by $\overline{S}_{CS}(A^\perp + Bdt) := 0$ otherwise. In the special case where $A^\perp \in \mathcal{A}^\perp \subset \mathcal{A}^\perp_{(\Sigma\backslash\{\sigma_0\})\times S^1}$ and $B \in C^\infty(\Sigma, \mathfrak{g}) \subset C^\infty(\Sigma\backslash\{\sigma_0\}, \mathfrak{g})$ Stokes' Theorem implies that

$$\int_{S^1} \ll \star dA^\perp(t), B \gg_{L^2_{\tilde{\mathfrak{g}}}(\Sigma, \mu_\mathbf{g})} dt = \ll A^\perp, \star dB \gg_{\mathcal{H}^\perp} \tag{3.6}$$

so $\overline{S}_{CS}$ is indeed an extension of $(S_{CS})_{|\mathcal{A}^{qax}}$. Moreover, it turns out that $\overline{S}_{CS}$ is a $\overline{\mathcal{G}_\Sigma}$-invariant function (for a detailed proof, see [24]). Thus we can apply Eq. (2.20) and obtain

$$\text{WLO}(L) \sim \sum_{\text{h}\in[\Sigma, G/T]} \int_{C^\infty(\Sigma, P)} \left[\int_{\mathcal{A}^\perp} \prod_i \text{Tr}_{\rho_i}\left(\mathcal{P}\exp\left(\int_{l_i} (\Omega_\text{h}^{-1} A^\perp \Omega_\text{h} + \Omega_\text{h}^{-1} d\Omega_\text{h} + Bdt)\right)\right)\right.$$

$$\times \exp(i\overline{S}_{CS}(\Omega_\text{h}^{-1} A^\perp \Omega_\text{h} + \Omega_\text{h}^{-1} d\Omega_\text{h} + Bdt)) DA^\perp \Big]$$

$$\times \triangle(B) \det\left(1_{\mathfrak{g}_0} - \exp(\text{ad}(B)_{|\mathfrak{g}_0})\right) DB \quad (3.7)$$

It is not difficult to see that with $A^\perp_{sing}(\text{h}) := \pi_\mathfrak{t}(\Omega_\text{h}^{-1} d\Omega_\text{h})$ we have

$$\overline{S}_{CS}(\Omega_\text{h}^{-1} A^\perp \Omega_\text{h} + \Omega_\text{h}^{-1} d\Omega_\text{h} + Bdt)$$
$$= \overline{S}_{CS}(\Omega_\text{h}^{-1} A^\perp \Omega_\text{h} + \pi_{\mathfrak{g}_0}(\Omega_\text{h}^{-1} d\Omega_\text{h}) + Bdt) + \frac{k}{2\pi} \ll \star dA^\perp_{sing}(\text{h}), B \gg \quad (3.8)$$

---

[6]for example, this will be the case if $A^\perp = \Omega_\text{h}^{-1} A_1^\perp \Omega_\text{h} + \Omega_\text{h}^{-1} d\Omega_\text{h}$ with $A_1^\perp \in \mathcal{A}^\perp$ and $\text{h} \in [\Sigma, G/T]$

where $\pi_{\mathfrak{g}_0}$ denotes the orthogonal projection $\mathfrak{g} \to \mathfrak{g}_0$ and where we have set

$$\ll \star dA^{\perp}_{sing}(\mathrm{h}), B \gg := \lim_{\epsilon \to 0} \int_{\Sigma \backslash B_\epsilon(\sigma_0)} \mathrm{Tr}(\star dA^{\perp}_{sing}(\mathrm{h})B) d\mu_{\mathbf{g}}$$

$$= \lim_{\epsilon \to 0} \int_{\Sigma \backslash B_\epsilon(\sigma_0)} \mathrm{Tr}(dA^{\perp}_{sing}(\mathrm{h}) \cdot B) \quad (3.9)$$

It is now tempting and, in fact, justified (cf. Sec. 4.2 in [24]) to make the change of variable $\Omega_{\mathrm{h}}^{-1} A^{\perp} \Omega_{\mathrm{h}} + \pi_{\mathfrak{g}_0}(\Omega_{\mathrm{h}}^{-1} d\Omega_{\mathrm{h}}) \longrightarrow A^{\perp}$. Note for example that, without loss of generality, we can assume that each mapping $\bar{g}_{\mathrm{h}} \in C^{\infty}(\Sigma \backslash \{\sigma_0\}, G/T)$ was chosen such that $\bar{g}_{\mathrm{h}} \equiv T \in G/T$ holds on a neighborhood U of the point σ_0. Then $(\Omega_{\mathrm{h}})_{|U}$ takes only values in T which implies $\pi_{\mathfrak{g}_0}(\Omega_{\mathrm{h}}^{-1} d\Omega_{\mathrm{h}}) = 0$ on U. So the 1-form $\pi_{\mathfrak{g}_0}(\Omega_{\mathrm{h}}^{-1} d\Omega_{\mathrm{h}})$ has no singularity in σ_0 and is therefore contained in $\mathcal{A}^{\perp}$. Thus we can replace $\Omega_{\mathrm{h}}^{-1} A^{\perp} \Omega_{\mathrm{h}} + \pi_{\mathfrak{g}_0}(\Omega_{\mathrm{h}}^{-1} d\Omega_{\mathrm{h}})$ by $\Omega_{\mathrm{h}}^{-1} A^{\perp} \Omega_{\mathrm{h}}$. Finally, it is also possible to make the change of variable $\Omega_{\mathrm{h}}^{-1} A^{\perp} \Omega_{\mathrm{h}} \longrightarrow A^{\perp}$ (taking into account that because, of the compactness of G, we have $\det(\mathrm{Ad}(\Omega_{\mathrm{h}}(\sigma))) = 1$ for every $\sigma \in \Sigma$; for more details see [24]). After this change of variable we arrive at the following equation

$$\mathrm{WLO}(L) \sim$$

$$\sum_{\mathrm{h}} \int_{C^{\infty}(\Sigma, P)} \left[\int_{\mathcal{A}^{\perp}} \prod_i \mathrm{Tr}_{\rho_i}\left(\mathcal{P} \exp\left(\int_{l_i} (A^{\perp} + A^{\perp}_{sing}(\mathrm{h}) + Bdt)\right)\right) \exp(iS_{CS}(A^{\perp} + Bdt)) DA^{\perp} \right]$$

$$\exp\left(i\tfrac{k}{2\pi} \ll \star dA^{\perp}_{sing}(\mathrm{h}), B \gg\right) \triangle[B] \det\left(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\right) DB \quad (3.10)$$

Remark 3.1. Note that the 1-forms $A^{\perp}_{sing}(\mathrm{h})$ are definitely not in $\mathcal{A}^{\perp}$ if $\mathrm{h} \neq [1_T]$. Thus it is not surprising that if one tries to make the additional change of variable $A^{\perp} + A^{\perp}_{sing}(\mathrm{h}) \longrightarrow A^{\perp}$ one obtains incorrect expressions, cf. again Sec. 4.2 in [24].

3.4. **The decomposition** $\mathcal{A}^{\perp} = \hat{\mathcal{A}}^{\perp} \oplus \mathcal{A}^{\perp}_c$. Let us now have a closer look at the informal measure $\exp(iS_{CS}(A^{\perp} + Bdt)) DA^{\perp}$ in Eq. (3.10) above. In view of Eq. (3.3) this measure is of "Gaussian type". Naively, one could try to identify its "mean" and "covariance operator" by writing down the following informal expression for $S_{CS}(A^{\perp} + Bdt)$, pretending that the operator $\frac{\partial}{\partial t} + \mathrm{ad}(B)$ in Eq. (3.3) is bijective:

$$S_{CS}(A^{\perp} + Bdt) = -\tfrac{k}{4\pi} \ll A^{\perp} - m(B), \star\left(\tfrac{\partial}{\partial t} + \mathrm{ad}(B)\right)(A^{\perp} - m(B)) \gg_{\mathcal{H}^{\perp}} \quad (3.11)$$

where $m(B) := (\frac{\partial}{\partial t} + \mathrm{ad}(B))^{-1} \cdot dB$. However, as the use of the word "pretend" above already indicates the problem with this naive ansatz is that the operator $\frac{\partial}{\partial t} + \mathrm{ad}(B) : \mathcal{A}^{\perp} \to \mathcal{A}^{\perp}$ is neither injective nor surjective so it is not clear what $(\frac{\partial}{\partial t} + \mathrm{ad}(B))^{-1}$ or $m(B) = (\frac{\partial}{\partial t} + \mathrm{ad}(B))^{-1} \cdot dB$ above should be.

In order to solve this problem let us first identify the kernel of $\frac{\partial}{\partial t} + \mathrm{ad}(B)$. It is easy to see that $\ker(\frac{\partial}{\partial t} + \mathrm{ad}(B)) = \mathcal{A}^{\perp}_c$ where

$$\mathcal{A}^{\perp}_c := \{A^{\perp} \in C^{\infty}(S^1, \mathcal{A}_{\Sigma}) \mid A^{\perp} \text{ is constant and } \mathcal{A}_{\Sigma,\mathfrak{t}}\text{-valued}\} \cong \mathcal{A}_{\Sigma,\mathfrak{t}} \quad (3.12)$$

(here we have used the identification $\mathcal{A}_{\Sigma} \cong \mathcal{A}_{\Sigma,\mathfrak{g}_0} \oplus \mathcal{A}_{\Sigma,\mathfrak{t}}$). So it is reasonable to introduce a direct sum decomposition of $\mathcal{A}^{\perp}$ of the form $\mathcal{A}^{\perp} = \mathcal{C} \oplus \mathcal{A}^{\perp}_c$ and then restrict $\frac{\partial}{\partial t} + \mathrm{ad}(B)$ to the space $\mathcal{C}$. This restriction is then clearly injective. A convenient choice for $\mathcal{C}$, already used in [23], is $\mathcal{C} := \hat{\mathcal{A}}^{\perp}$ where

$$\hat{\mathcal{A}}^{\perp} := \{A^{\perp} \in C^{\infty}(S^1, \mathcal{A}_{\Sigma}) \mid A^{\perp}(t_0) \in \mathcal{A}_{\Sigma,\mathfrak{g}_0}\} \quad (3.13)$$

Note that the operator $\frac{\partial}{\partial t} + \mathrm{ad}(B)$, when restricted onto $\hat{\mathcal{A}}^{\perp}$, is still not surjective. We solve this problem by replacing $\hat{\mathcal{A}}^{\perp}$ by the slightly bigger space[7]

$$\tilde{\mathcal{A}}^{\perp} := \hat{\mathcal{A}}^{\perp} \oplus \{A^{\perp}_c \cdot (i^{-1}_{t_0}(\cdot) - 1/2) \mid A^{\perp}_c \in \mathcal{A}_{\Sigma,\mathfrak{t}}\} \quad (3.14)$$

---

[7]in Sec. 8 in [23] we gave a detailed motivation for this ansatz in the special case $t_0 = i_{S^1}(0)$ (note that the space $\tilde{\mathcal{A}}^{\perp}$ was denoted by $\tilde{C}^{\infty}(S^1, \mathcal{A}_{\Sigma})$ there)

where $i_{t_0}^{-1}$ is the inverse of the bijection

$$i_{t_0} : [0, 1) \ni s \mapsto i_{S^1}(s) \cdot t_0 \in S^1 \tag{3.15}$$

(here i_{S^1} is the mapping defined at the beginning of Sec. 2 and "$\cdot$" denotes the standard multiplication of $S^1 \subset \mathbb{C}$).

We can now extend $\frac{\partial}{\partial t} : \mathcal{A}^\perp \to \mathcal{A}^\perp$ in an obvious way to an operator $\tilde{\mathcal{A}}^\perp \to \mathcal{A}^\perp$ and it turns out that the (extended) operator $(\frac{\partial}{\partial t} + \mathrm{ad}(B)) : \tilde{\mathcal{A}}^\perp \to \mathcal{A}^\perp$ is a bijection for every $B \in C^\infty(\Sigma, P)$. The inverse operator $(\frac{\partial}{\partial t} + \mathrm{ad}(B))^{-1} : \mathcal{A}^\perp \to \tilde{\mathcal{A}}^\perp$ is given explicitly by

$$\forall t \in S^1 : \quad \left((\tfrac{\partial}{\partial t} + \mathrm{ad}(B))^{-1} A^\perp\right)(t) = \frac{1}{2}\left[\int_0^{i_{t_0}^{-1}(t)} A^\perp(i_{t_0}(s))ds - \int_{i_{t_0}^{-1}(t)}^1 A^\perp(i_{t_0}(s))ds \right] \tag{3.16a}$$

if[8] $A^\perp \in C^\infty(S^1, \mathcal{A}_\Sigma)$ takes only values in $\mathcal{A}_{\Sigma,\mathfrak{t}}$ and

$$\forall t \in S^1 : \quad \left((\tfrac{\partial}{\partial t} + \mathrm{ad}(B))^{-1} A^\perp\right)(t)$$
$$= \left(\exp(\mathrm{ad}(B)_{|\mathfrak{g}_0}) - 1_{\mathfrak{g}_0}\right)^{-1} \cdot \int_0^1 \exp(s \cdot \mathrm{ad}(B)) A^\perp(i_{S^1}(s) \cdot t)ds \tag{3.16b}$$

if $A^\perp \in C^\infty(S^1, \mathcal{A}_\Sigma)$ takes only values in $\mathcal{A}_{\Sigma,\mathfrak{g}_0}$. Note that the last expression is well-defined because each $B(\sigma)$, $\sigma \in \Sigma$, is an element of the (open) alcove P, from which it follows that $\exp(\mathrm{ad}(B(\sigma))_{|\mathfrak{g}_0}) - \mathrm{id}_{\mathfrak{g}_0} \in \mathrm{End}(\mathfrak{g}_0)$ is invertible, cf. Remark 8.1 in [23].

We can now define $m(B)$ rigorously by

$$m(B) := (\tfrac{\partial}{\partial t} + \mathrm{ad}(B))^{-1} \cdot dB \overset{(*)}{=} (i_{t_0}^{-1}(\cdot) - 1/2) \cdot dB \in \tilde{\mathcal{A}}^\perp \tag{3.17}$$

(here step $(*)$ follows from Eq. (3.16a)). With this definition we have

$$S_{CS}(\hat{A}^\perp + Bdt) = -\frac{k}{4\pi} \ll \hat{A}^\perp - m(B), \star(\tfrac{\partial}{\partial t} + \mathrm{ad}(B))(\hat{A}^\perp - m(B)) \gg_{\mathcal{H}^\perp} \tag{3.18}$$

for all $\hat{A}^\perp \in \hat{\mathcal{A}}^\perp$ and $B \in C^\infty(\Sigma, P)$. Moreover, we have

$$S_{CS}(\hat{A}^\perp + A_c^\perp + Bdt) = S_{CS}(\hat{A}^\perp + Bdt) - \frac{k}{2\pi} \ll A_c^\perp, \star dB \gg_{\mathcal{H}_\Sigma} \tag{3.19}$$

In Eq. (3.10), the informal measure "$\exp(iS_{CS}(A^\perp + Bdt)DA^\perp$" appeared as part of a multiple integral. According to Eq. (3.19) we can write $\exp(iS_{CS}(A^\perp + Bdt)DA^\perp$ in the form $(\exp(iS_{CS}(\hat{A}^\perp + Bdt))D\hat{A}^\perp) \otimes (\exp(-i\frac{k}{2\pi} \ll A_c^\perp, \star dB \gg_{\mathcal{H}_\Sigma}))DA_c^\perp)$ and according to Eq. (3.18) the first factor is, at an informal level, a "Gauss-type" measure with "mean" m(B), "covariance operator" $C(B) : \mathcal{A}^\perp \to \tilde{\mathcal{A}}^\perp$ given by

$$C(B) := -\tfrac{2\pi i}{k}(\tfrac{\partial}{\partial t} + \mathrm{ad}(B))^{-1} \circ \star^{-1} \tag{3.20}$$

Let us now plug in the decomposition $\mathcal{A}^\perp = \hat{\mathcal{A}}^\perp \oplus \mathcal{A}_c^\perp$ into Eq. (3.10) above. Taking into account Eqs. (3.18), (3.19) and the equality $\ll \star dA_c^\perp, B \gg_{L_\mathfrak{g}^2(\Sigma, \mu_\mathbf{g})} = \ll A_c^\perp, \star dB \gg_{\mathcal{H}_\Sigma}$ we obtain

$\mathrm{WLO}(L) \sim$

$$\sum_h \int_{\mathcal{A}_c^\perp \times C^\infty(\Sigma, P)} \left[\int_{\hat{\mathcal{A}}^\perp} \prod_i \mathrm{Tr}_{\rho_i}\left(\mathcal{P}\exp\left(\int_{l_i}(\hat{A}^\perp + A_c^\perp + A_{sing}^\perp(\mathrm{h}) + Bdt)\right)\right)d\hat{\mu}_B^\perp(\hat{A}^\perp) \right]$$
$$\times \left\{ \exp(i\tfrac{k}{2\pi} \ll \star dA_{sing}^\perp(\mathrm{h}), B \gg)\det\left(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\right)\triangle[B]\hat{Z}(B) \right\}$$
$$\times \exp(i\tfrac{k}{2\pi} \ll \star dA_c^\perp, B \gg_{L_\mathfrak{t}^2(\Sigma, d\mu_\mathbf{g})})(DA_c^\perp \otimes DB) \tag{3.21}$$

---

[8]note that in this case $(\frac{\partial}{\partial t} + \mathrm{ad}(B))^{-1} \cdot A^\perp = (\frac{\partial}{\partial t})^{-1} \cdot A^\perp$ so it is clear that the right-hand side of Eq. (3.16a) can not depend on B

where

$$\hat{Z}(B) := \int \exp(iS_{CS}(\hat{A}^\perp + Bdt))D\hat{A}^\perp \tag{3.22}$$

$$d\hat{\mu}_B^\perp(\hat{A}^\perp) := \frac{1}{\hat{Z}(B)} \exp(iS_{CS}(\hat{A}^\perp + Bdt))D\hat{A}^\perp \tag{3.23}$$

Note that

$$\hat{Z}(B) \sim |\det(\tfrac{\partial}{\partial t} + \mathrm{ad}(B))|^{-1/2} \tag{3.24}$$

3.5. **Evaluation of** $\det\big(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\big)\triangle[B]\hat{Z}(B)$. Naively, one might expect that for $B \in C^\infty(\Sigma, P)$ we have

$$\triangle[B]\hat{Z}(B) \sim \frac{|\det(\tfrac{\partial}{\partial t} + \mathrm{ad}(B))|}{|\det(\tfrac{\partial}{\partial t} + \mathrm{ad}(B))|^{1/2}} = 1 \tag{3.25}$$

where the operator $\frac{\partial}{\partial t} + \mathrm{ad}(B)$ in the numerator is defined on $C_{\mathfrak{g}}^\infty(\Sigma \times S^1)$ and the operator in the denominator is defined on $C^\infty(S^1, \mathcal{A}_\Sigma)$.

However, the detailed analysis in Sec. 6 of [10] suggests that, already in the simplest case, i.e. the case of constant[9] $B \equiv b$, $b \in P$, the expression

$$\det\big(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\big)\triangle[B]\hat{Z}(B) \tag{3.26}$$

should be replaced by the more complicated expression

$$\big(\det\big(\mathrm{id}_{\mathfrak{g}_0} - \exp(\mathrm{ad}(b)_{|\mathfrak{g}_0})\big)\big)^{\chi(\Sigma)/2} \times \exp\big(i\tfrac{c_G}{2\pi} \ll \star dA_c^\perp + \star dA_{sing}^\perp(\mathrm{h}), b \gg_{L^2_{\mathfrak{t}}(\Sigma, d\mu_{\mathbf{g}})}\big) \tag{3.27}$$

where c_G is the dual Coxeter number[10] of G.

In Subsec. 6.3 below, not only constant functions B will appear but more general "step functions", i.e. functions B which are constant on each of the connected components $X_1, X_2, \ldots, X_\mu$ of the set $\Sigma \backslash (\bigcup_j \mathrm{arc}(l_\Sigma^j))$, cf. Subsec. 3.1.

Remark 3.2. Of course, these "step functions" are not well-defined elements of $C^\infty(\Sigma, P)$. Thus it is actually necessary to use an additional regularization procedure in Subsec. 6.3 by which the step functions are replaced by certain smooth approximations (later one has to perform a limit procedure). As the implementation of this additional regularization procedure is on one hand straightforward and, on the other hand, would give rise to some rather clumsy notation which would distract the reader from the main line of argument of this paper we have decided not to include this additional regularization procedure here but to postpone it to a subsequent paper.

The expression (3.27) and the results that we will obtain in Subsec. 6.3 below strongly suggest that for such "step functions" B the expression (3.26) should be replaced by

$$\prod_{t=1}^\mu \big(\det\big(\mathrm{id}_{\mathfrak{g}_0} - \exp(\mathrm{ad}(b_t)_{|\mathfrak{g}_0})\big)\big)^{\chi(X_t)/2} \times \exp\big(i\tfrac{c_G}{2\pi} \ll \star dA_c^\perp + \star dA_{sing}^\perp(\mathrm{h}), B \gg_{L^2_{\mathfrak{t}}(\Sigma, d\mu_{\mathbf{g}})}\big) \tag{3.28}$$

where $b_t \in P$, $t \leq \mu$, are given by $B_{|X_t} \equiv b_t$.

If we want to work with Eq. (3.21) we have to make sense of (3.26) for all $B \in C^\infty(\Sigma, P)$ even if later only special B will play a role. In view of (3.28) we suggest that for general $B \in C^\infty(\Sigma, P)$ the expression (3.26) should be replaced by the (metric dependent) expression (cf. Remark 3.3)

$$\det_{reg}\big(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\big) \times \exp\big(i\tfrac{c_G}{2\pi} \ll \star dA_c^\perp + \star dA_{sing}^\perp(\mathrm{h}), B \gg_{L^2_{\mathfrak{t}}(\Sigma, d\mu_{\mathbf{g}})}\big) \tag{3.29}$$

---

[9]which is the only case of relevance in [10], cf. our Subsec. 6.2 below

[10]for example, for $G = SU(N)$ we have $c_G = N$. This gives rise to the "charge shift" $k \to k + N$

where

$$\det{}_{reg}\big(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\big) :=$$

$$\prod_{t=1}^{\mu} \exp\bigg(\tfrac{1}{vol(X_t)} \int_{X_t} \ln\big(\det\big(\mathrm{id}_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B(\sigma))_{|\mathfrak{g}_0})\big)\big) d\mu_{\mathbf{g}}(\sigma) \bigg)^{\chi(X_t)/2} \quad (3.30)$$

With this Ansatz we finally arrive at the following heuristic formula for the WLOs which will be fundamental for the rest of this paper.

$$\mathrm{WLO}(L) \sim$$

$$\sum_{\mathrm{h}\in[\Sigma, G/T]} \int_{\mathcal{A}_c^{\perp} \times \mathcal{B}} \bigg[\int_{\hat{\mathcal{A}}^{\perp}} \prod_i \mathrm{Tr}_{\rho_i}\big(\mathcal{P}\exp(\int_{l_i} (\hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + B dt)))d\hat{\mu}_B^{\perp}(\hat{A}^{\perp}) \bigg]$$

$$\times \big\{ \exp(i\tfrac{k+c_G}{2\pi} \ll \star dA_{sing}^{\perp}(\mathrm{h}), B \gg) \det{}_{reg}\big(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\big) \big\}$$

$$\times \exp\big(i\tfrac{k+c_G}{2\pi} \ll \star dA_c^{\perp}, B \gg_{L_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathbf{g}})}\big)(DA_c^{\perp} \otimes DB) \quad (3.31)$$

where

$$\mathcal{B} := C^{\infty}(\Sigma, P) \qquad\qquad (3.32)$$

Eq. (3.31) can be considered to be the generalization of formula (7.1) in [10] to arbitrary links (cf. also Sec. 7.6 in [10]).

Remark 3.3. It would be desirable to find a more thorough justification (which is independent of the considerations in Subsec. 6.3 below) for replacing expression (3.26) by (3.29). In particular, such a justification will have to explain/answer why – for making sense of the expression (3.26) – one has to use a regularization scheme that depends on the link L even though the expression (3.26) does not, cf. Remark 4.2 in [24] for some more details.

3.6. The explicit Computation of the WLOs: overview.
We will divide the evaluation of the right-hand side of Eq. (3.31) into the following three steps:

- **Step 1:** Make sense of the integral functional $\int \cdots d\hat{\mu}_B^{\perp}$
- **Step 2:** Make sense of the "inner" integral $\int_{\hat{\mathcal{A}}^{\perp}} \prod_i \mathrm{Tr}_{\rho_i}(\mathcal{P}\exp(\int_{l_i} \hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + B dt))d\hat{\mu}_B^{\perp}(\hat{A}^{\perp})$ in Eq. (3.31) and compute its value.
- **Step 3:** Make sense of the total expression on the right-hand side of Eq. (3.31) and compute its value.

4. The Computation of the WLOs: Step 1

In Sec. 8 in [23] we gave a rigorous implementation $\Phi_B^{\perp}$ of the integral functional $\int \cdots d\hat{\mu}_B^{\perp}$. Here we briefly recall the construction of $\Phi_B^{\perp}$. Eqs. (3.18), (3.19) (3.17), and (3.20) suggest that the heuristic "measure" $\hat{\mu}_B^{\perp}$ on $\hat{\mathcal{A}}^{\perp}$ is of "Gaussian type" with "mean" $m(B)$ and "covariance operator" $C(B)$. From the results in Sec. 8 in [23] it follows[11] that the operator $C(B)$: $\mathcal{A}^{\perp} \to \tilde{\mathcal{A}}^{\perp} \subset \mathcal{H}^{\perp}$ is a bounded and symmetric (densely defined) operator on $\mathcal{H}^{\perp} = L_{\mathcal{H}_{\Sigma}}^2(S^1, dt)$. This allows us to use the standard approach of white noise analysis and to realize the integral functional $\int \cdots d\hat{\mu}_B^{\perp}$ rigorously as a generalized distribution $\Phi_B^{\perp}$ on the topological dual $\mathcal{N}^*$ of a suitably chosen nuclear subspace $\mathcal{N}$ of $\mathcal{H}^{\perp}$. We will not go into details here. Let us mention here only the following points:

i) It turned out in [23] that the nuclear space $\mathcal{N}$ which was chosen there using a standard procedure coincides with the space $\mathcal{A}^{\perp}$. Thus the operator $C(B)$ can be considered to be an operator $\mathcal{N} \to \mathcal{H}^{\perp}$.

---

[11]Note that the Hilbert space $\mathcal{H}$ appearing in Subsec. 8.2. in [23] is naturally isomorphic to our Hilbert space $\mathcal{H}^{\perp}$ and we can therefore identify the two Hilbert spaces with each other

ii) The statement that $\Phi_B^\perp$ is a generalized distribution $\mathcal{N}^*$ means that $\Phi_B^\perp$ is a continuous linear functional $(\mathcal{N}) \to \mathbb{C}$ where the topological space $(\mathcal{N})$ ("the space of test functions") is defined in a suitable way. We will not give a full definition of $(\mathcal{N})$ here as this is rather technical. For our purposes it is enough to know that each test function $\psi \in (\mathcal{N})$ is a continuous mapping $\mathcal{N}^* \to \mathbb{C}$ and that $(\mathcal{N})$ contains the trigonometric exponentials $\exp(i(\cdot, j)) : \mathcal{N}^* \to \mathbb{C}$, $j \in \mathcal{N}$, and the polynomial functions $\prod_{i=1}^{n} (\cdot, j_i) : \mathcal{N}^* \to \mathbb{C}$, $j_1, j_2, \ldots, j_n \in \mathcal{N}$. Here $(\cdot, \cdot) : \mathcal{N}^* \times \mathcal{N} \to \mathbb{R}$ denotes the canonical pairing.

iii) The generalized distribution $\Phi_B^\perp$ was defined in [23] as the unique continuous linear functional $(\mathcal{N}) \to \mathbb{C}$ with the property that

$$\Phi_B^\perp(\exp(i(\cdot, j))) = \exp(i \ll j, m(B) \gg_{\mathcal{H}^\perp}) \exp(-\tfrac{1}{2} \ll j, C(B)j \gg_{\mathcal{H}^\perp}) \tag{4.1}$$

holds for all $j \in \mathcal{N}$. Note that $\Phi_B^\perp(\exp(i(\cdot, j)))$ is the analogue of the Fourier transformation of the Gauss-type "measure" $\hat{\mu}_B^\perp$ and, at a heuristic level, one expects that this Fourier transform is indeed given by the right-hand side of Eq. (4.1).

iv) The "moments" of $\Phi_B^\perp$, i.e. the expressions $\Phi_B^\perp(\prod_{i=1}^{n}(\cdot, j_i))$ with fixed $j_1, j_2, \ldots, j_n \in \mathcal{N}$ can be computed easily, using similar arguments as in the proof of Proposition 3 in [22]. In particular, the first and second moments are given by

$$\Phi_B^\perp((\cdot, j_1)) = \ll j_1, m(B) \gg_{\mathcal{H}^\perp} \tag{4.2}$$

and

$$\Phi_B^\perp((\cdot, j_1) \cdot (\cdot, j_2)) = \ll j_1, C(B) \, j_2 \gg_{\mathcal{H}^\perp} + \ll j_1, m(B) \gg_{\mathcal{H}^\perp} \cdot \ll j_2, m(B) \gg_{\mathcal{H}^\perp} \tag{4.3}$$

for all $j_1, j_2 \in \mathcal{N}$.

The higher moments are given by expressions that are totally analogous to the expressions that appear in the classical Wick theorem for the moments of a Gaussian probability measure on a Euclidean space.

v) Clearly, the linear functional $\Phi_B^\perp : (\mathcal{N}) \to \mathbb{C}$ induces a linear function $(\mathcal{N}) \otimes_{\mathbb{C}} \mathrm{Mat}(N, \mathbb{C}) \to \mathrm{Mat}(N, \mathbb{C})$ in an obvious way, which will also be denoted by $\Phi_B^\perp$.

5. The Computation of the WLOs: Step 2

In order to make sense of $\int_{\hat{\mathcal{A}}^\perp} \prod_i \mathrm{Tr}_{\rho_i}(\mathcal{P} \exp(\int_{l_i} \hat{A}^\perp + A_c^\perp + A_{sing}^\perp(\mathrm{h}) + Bdt)) d\hat{\mu}_B^\perp(\hat{A}^\perp)$ we proceed in the following way:

- We regularize $\prod_i \mathrm{Tr}_{\rho_i}(\mathcal{P} \exp(\int_{l_i} \hat{A}^\perp + A_c^\perp + A_{sing}^\perp(\mathrm{h}) + Bdt))$ by using "smeared loops" l_i^ϵ. Later we let $\epsilon \to 0$.
- Then we introduce "deformations" $\Phi_{B,\phi_s}^\perp$ of $\Phi_B^\perp$ w.r.t. a suitable family $(\phi_s)_{s>0}$ of diffeomorphisms of $\Sigma \times S^1$ such that $\phi_s \to \mathrm{id}_{\Sigma \times S^1}$ uniformly as $s \to 0$ ("Framing")
- Finally we prove that the limit[12]

$$\mathrm{WLO}(L, \phi_s; A_c^\perp, A_{sing}^\perp(\mathrm{h}), B) :=$$

$$\lim_{\epsilon \to 0} \Phi_{B,\phi_s}^\perp \left(\prod_i \mathrm{Tr}_{\rho_i}(\mathcal{P} \exp(\int_{l_i^\epsilon} (\cdot) + A_c^\perp + A_{sing}^\perp(\mathrm{h}) + Bdt)) \right) \tag{5.1}$$

exists and we compute this limit explicitly for small $s > 0$.

5.1. **Abelian G and general L.** Let us start with considering the case where G is Abelian, i.e. a torus, and where $\Sigma = S^2$. For Abelian G we have $G = T$, $P = \mathfrak{t}$, $\mathfrak{g}_0 = \{0\}$, $c_G = 0$, and $[\Sigma, G/T] = \{[1_T]\}$. Thus we can drop the expression $\det_{reg}(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0}))$ and we can

---

[12]for Abelian G we have $A_{sing}^\perp(\mathrm{h}) = 0$ for $\mathrm{h} \in [\Sigma, G/T] = \{[1_T]\}$ so in this case we will use the notation $\mathrm{WLO}(L, \phi_s; A_c^\perp, B)$ instead of $\mathrm{WLO}(L, \phi_s; A_c^\perp, A_{sing}^\perp(\mathrm{h}), B)$

choose $\Omega_h = 1$, for $h = [1_T]$, from which $A^\perp_{sing}(h) = 0$ follows. Accordingly, Eq. (3.31) simplifies and we obtain

$$\text{WLO}(L) \sim \int_{\mathcal{A}^\perp_c \times C^\infty(\Sigma, \mathfrak{t})} \left[\int_{\hat{A}^\perp} \prod_i \text{Tr}_{\rho_i}\big(\mathcal{P}\exp\big(\int_{l_i} (\hat{A}^\perp + A^\perp_c + Bdt)\big)\big) d\hat{\mu}^\perp_B(\hat{A}^\perp) \right]$$

$$\times \exp(i\tfrac{k}{2\pi} \ll \star dA^\perp_c, B \gg_{L^2_\mathfrak{t}(\Sigma, d\mu_\mathbf{g})})(DA^\perp_c \otimes DB) \quad (5.2)$$

For simplicity, we will only consider the special case where $G = U(1)$ and where every ρ_i is equal to the fundamental representation $\rho_{U(1)}$ of $U(1)$. In this case we can choose the basis $(T_a)_{a \leq \dim(G)}$ to consist of the single element $T_1 = i \in u(1)$. Clearly, we have

$$\text{Tr}_{\rho_{U(1)}}(\mathcal{P}\exp(\int_l \hat{A}^\perp + A^\perp_c + Bdt)) = \exp(\int_l \hat{A}^\perp)\exp(\int_l A^\perp_c)\exp(\int_l Bdt)$$

for every loop l.

Let us now replace in Eq. (5.2) the integral functional $\int \cdots d\hat{\mu}^\perp_B$ by the functional $\Phi^\perp_B$ which we have introduce in Sec. 4. As we pointed out in Sec. 4, $\Phi^\perp_B$ is a generalized distribution on the topological dual $\mathcal{N}^*$ of $\mathcal{N} = \mathcal{A}^\perp$. A general element $\hat{A}^\perp \in \mathcal{N}^*$ will not be a smooth function, so $\int_l \hat{A}^\perp = \int_0^1 \hat{A}^\perp(l'(s))ds$ does not make sense in general. In [23] we solved this problem by replacing $\hat{A}^\perp(l'(s))$, $s \in [0,1]$, by $T_1(\hat{A}^\perp, f_1^{l_\epsilon}(s))$, $\epsilon > 0$, for suitable elements $f_1^{l_\epsilon}(s)$ of $\mathcal{N} = \mathcal{A}^\perp$ which were defined using parallel transport w.r.t. the Levi-Civita connection of $(\Sigma, \mathbf{g})$ (here $(\cdot, \cdot)$ denotes again the canonical pairing $\mathcal{N}^* \times \mathcal{N} \to \mathbb{R}$). However, this Ansatz requires the use of some rather clumsy notation which distracts from the main points of the computation. For this reason we will proceed in a different way in the present paper. Here we will just concentrate on the special situation when the following condition is fulfilled:

(S) There is an open subset U of Σ which is diffeomorphic to $\mathbb{R}^2$ and which "contains" all the l^j_Σ, i.e. which fulfills $\text{arc}(l^j_\Sigma) \subset U$, $j \leq n$.

In this case U inherits an group structure from $\mathbb{R}^2$ and we can then use this group structure $+ : U \times U \to U$ rather than parallel transport w.r.t. the Levi-Civita connection for the definition of the functions $f_1^{l_\epsilon}(s)$, $s \in [0,1]$, $\epsilon > 0$. Moreover, by "identifying" U with $\mathbb{R}^2$ we can simplify our notation.

Let us fix a Dirac family $(\delta^\epsilon_{S^1})_{\epsilon>0}$ on S^1 in the point $1 = i_{S^1}(0) \in S^1$ and a Dirac family $(\delta^\epsilon_\Sigma)_{\epsilon>0}$ on U in the point $(0,0) \in U \cong \mathbb{R}^2$. Then we obtain a Dirac family $(\delta^\epsilon)_{\epsilon>0}$ on $U \times S^1$ (and thus also on $\Sigma \times S^1$) in the point $((0,0),1)$ given by $\delta^\epsilon(\sigma, t) = \delta^\epsilon_\Sigma(\sigma)\delta^\epsilon_{S^1}(t)$. We now define $f_1^{l_\epsilon}(s) \in \mathcal{A}^\perp_{U \times S^1}$ by $f_1^{l_\epsilon}(s) = T_1 l'_\Sigma(s)\delta^\epsilon(\cdot - l(s))$ where we have used the identification $\mathcal{A}^\perp_{U \times S^1} \cong C^\infty(U \times S^1, \mathbb{R}^2 \otimes \mathfrak{g})$ (induced by the identification $U \cong \mathbb{R}^2$) and where "$-$" denotes the subtraction associated to the product group structure $+ : (U \times S^1) \times (U \times S^1) \to U \times S^1$. As $f_1^{l_\epsilon}(s)$ has compact support and as the subspace of $\mathcal{A}^\perp_{U \times S^1}$ which consists of all elements with compact support can be embedded naturally into the space $\mathcal{A}^\perp$ we can consider $f_1^{l_\epsilon}(s)$ as an element of $\mathcal{A}^\perp$. Instead of using the notation $f_1^{l_\epsilon}(s)$ we will use the more suggestive notation $T_1 l'_\Sigma(s)\delta^\epsilon(\cdot - l(s))$ in the sequel and we set for every $\hat{A}^\perp \in \mathcal{N}^*$

$$\int_{l^\epsilon_i} \hat{A}^\perp := T_1(\hat{A}^\perp, T_1 \int_0^1 (l^i_\Sigma)'(s)\delta^\epsilon(\cdot - l_i(s))ds),$$

$$\mathcal{P}\exp\big(\int_{l^\epsilon_i} (\hat{A}^\perp + A^\perp_c + Bdt)\big) := \exp(\int_{l^\epsilon_i} \hat{A}^\perp)\exp(\int_{l_i} A^\perp_c)\exp(\int_{l_i} Bdt) \quad (5.3)$$

and

$$\text{WLO}(L; A^\perp_c, B) := \lim_{\epsilon \to 0} \Phi^\perp_B\Big(\prod_i \text{Tr}_{\rho_i}(\mathcal{P}\exp(\int_{l^\epsilon_i} (\cdot) + A^\perp_c + Bdt))\Big) \quad (5.4)$$

provided that the limit on the right-hand side exists.

Remark 5.1. Informally, we have $\int_l \hat{A}^\perp = \int_0^1 \hat{A}^\perp(l'(s))ds = T_1(\hat{A}^\perp, T_1 \int_0^1 l'_\Sigma(s)\delta(\cdot - l(s))ds)$ where δ denotes the informal "Dirac function" on $U \times S^1$ in the point $((0,0),1)$. So "loop smearing" just amounts to replacing the ill-defined expression $\delta(\cdot - l(s))$ by the test function $\delta^\epsilon(\cdot - l(s))$

If we insert Eq. (5.3) into Eq. (5.4) above we obtain

$$\text{WLO}(L; A_c^\perp, B) = \prod_j \exp(\int_{l_j} A_c^\perp) \exp(\int_{l_j} B dt)$$
$$\times \lim_{\epsilon \to 0} \Phi_B^\perp \Big(\exp\big(T_1(\cdot, \sum_i T_1 \int_0^1 (l_\Sigma^i)'(s)\delta^\epsilon(\cdot - l_i(s))ds)\big) \Big) \quad (5.5)$$

From $T_1 = i$ and Eq. (4.1) we obtain

$$\Phi_B^\perp \Big(\exp\big(T_1(\cdot, \sum_i T_1 \int_0^1 (l_\Sigma^i)'(s)\delta^\epsilon(\cdot - l_i(s))ds)\big) \Big)$$
$$= \prod_{j,k} \exp(-\frac{1}{2} \ll T_1 \int_0^1 (l_\Sigma^j)'(s)\delta^\epsilon(\cdot - l_j(s))ds, C(B) \cdot \big(T_1 \int_0^1 (l_\Sigma^k)'(u)\delta^\epsilon(\cdot - l_k(u))du\big) \gg_{\mathcal{H}^\perp})$$
$$\times \prod_i \exp(T_1 \ll m(B), T_1 \int_0^1 (l_\Sigma^i)'(t)\delta^\epsilon(\cdot - l_i(t))dt \gg_{\mathcal{H}^\perp}) \quad (5.6)$$

Clearly, we have

$$\lim_{\epsilon \to 0} T_1 \ll m(B), T_1 \int_0^1 (l_\Sigma^i)'(t)\delta^\epsilon(\cdot - l_i(t))dt \gg_{\mathcal{H}^\perp}$$
$$= \int_0^1 (i_{t_0}^{-1}(l_{S^1}^i(t)) - 1/2)dB((l_\Sigma^i)'(t))dt = \int_0^1 l_{\mathbb{R}}^i(t)\frac{d}{dt}B(l_\Sigma^i(t))dt \quad (5.7)$$

where we have set $l_{\mathbb{R}}^i := i_{t_0}^{-1} \circ l_{S^1}^i - 1/2$. Thus we obtain from Eqs. (5.4)–(5.7)

$$\text{WLO}(L; A_c^\perp, B) = \big(\prod_{j,k} \exp(-\frac{1}{2}T(l_j, l_k))\big)\big(\prod_j \exp(\int_{l_j} A_c^\perp)\big)$$
$$\times \{\prod_j \exp(\int l_{\mathbb{R}}^j(t)\frac{d}{dt}B(l_\Sigma^j(t))dt) \exp(\int_{l_j} B dt)\} \quad (5.8)$$

where we have set

$$T(l_j, l_k) :=$$
$$\lim_{\epsilon \to 0} \ll T_1 \int_0^1 (l_\Sigma^j)'(s)\delta^\epsilon(\cdot - l_j(s))ds, C(B) \cdot \big(T_1 \int_0^1 (l_\Sigma^k)'(u)\delta^\epsilon(\cdot - l_k(u))du\big) \gg_{\mathcal{H}^\perp}) \quad (5.9)$$

provided that the limit $T(l_j, l_k)$ exists for each pair (l_j, l_k). Taking into account that $(l_{S^1}^j)'(t) = (l_{\mathbb{R}}^j)'(t)$ we see that

$$\int l_{\mathbb{R}}^j(t)\frac{d}{dt}B(l_\Sigma^j(t))dt + \int_{l_j} B dt = \int_0^1 \{l_{\mathbb{R}}^j(u)\frac{d}{du}B(l_\Sigma^j(u)) + B(l_\Sigma^j(u)) \cdot (l_{\mathbb{R}}^j)'(u)\}du$$
$$= \sum_{i=0}^{n_j+1} \int_{s_i^j}^{s_{i+1}^j} \frac{d}{du}\big[l_{\mathbb{R}}^j(u) \cdot B(l_\Sigma^j(u))\big]du = \sum_{i=1}^{n_j} \text{sgn}(l_{S^1}^j; s_i^j) \cdot B(l_\Sigma^j(s_i^j)) \quad (5.10)$$

where $(s_i^j)_{0 \le i \le n_j+1}$ denotes the strictly increasing sequence of $[0,1]$ given by $s_0^j := 0$, $s_{n_j+1}^j = 1$, and $\{s_i^j \mid 1 \le i \le n_j\} = I_j(t_0) := (l_{S^1}^j)^{-1}(\{t_0\})$ with $n_j := \#I_j(t_0)$ and where[13] we have set

---

[13] in the special case where $0 \in I_j(t_0)$ the definition of $\text{sgn}(l_{S^1}^j; s_i^j)$ has to be modified in the obvious way

$\operatorname{sgn}(l^j_{S^1}; s^j_i) := \lim_{s \uparrow s^j_i} l^j_{\mathbb{R}}(s) - \lim_{s \downarrow s^j_i} l^j_{\mathbb{R}}(s) \in \{-1, 0, 1\}$. Thus it follows that the last factor in Eq. (5.8) equals

$$\prod_j \exp\Big(\sum_{u \in I_j(t_0)} \operatorname{sgn}(l^j_{S^1}; u) \cdot B(l^j_{\Sigma}(u)) \Big) \tag{5.11}$$

Let us now evaluate the expression $T(l_j, l_k)$ for fixed $j, k \leq n$. We will first concentrate on the case where $j \neq k$. As $C(B) = -\frac{2\pi i}{k}(\frac{\partial}{\partial t} + \operatorname{ad}(B))^{-1} \circ \star^{-1} = -\frac{2\pi i}{k}(\frac{\partial}{\partial t})^{-1} \circ (-\star) = \frac{2\pi i}{k} \star \circ (\frac{\partial}{\partial t})^{-1}$ we obtain from Eq. (3.16a), setting $l := l_j$, $\tilde{l} := l_k$,

$$T(l, \tilde{l}) = \frac{2\pi i}{k} \lim_{\epsilon \to 0} \int_0^1 \int_0^1 \Bigg[\int_{S^1} \delta^\epsilon_{S^1}(t - l_{S^1}(s)) \big(\frac{\partial}{\partial t}^{-1} \delta^\epsilon_{S^1}(t - \tilde{l}_{S^1}(u)) \big) \ dt$$

$$\times \ll T_1 l'_{\Sigma}(s) \delta^\epsilon_{\Sigma}(\cdot - l_{\Sigma}(s)), \star (T_1 \tilde{l}'_{\Sigma}(u) \delta^\epsilon_{\Sigma}(\cdot - \tilde{l}_{\Sigma}(u))) \gg_{\mathcal{H}_\Sigma} \Bigg] ds \ du \tag{5.12}$$

where $\frac{\partial}{\partial t}^{-1}$ denotes the operator given by $\big(\frac{\partial}{\partial t}^{-1} f \big)(t) = \frac{1}{2} \big[\int_0^{i^{-1}_{t_0}(t)} f(i_{S^1}(s) \cdot t_0) ds - \int_{i^{-1}_{t_0}(t)}^1 f(i_{S^1}(s) \cdot t_0) ds \big]$ for all $f \in C^\infty(S^1, \mathbb{R})$ and $t \in S^1$. Clearly, for fixed $s, u \in [0, 1]$ and sufficiently small ϵ we have

$$\int_{S^1} \delta^\epsilon_{S^1}(t - l_{S^1}(s)) \ \big(\frac{\partial}{\partial t}^{-1} \delta^\epsilon_{S^1}(t - \tilde{l}_{S^1}(u)) \big) \ dt \ = \ -\frac{1}{2} \big[1_{l_{S^1}(s) < \tilde{l}_{S^1}(u)} - 1_{l_{S^1}(s) > \tilde{l}_{S^1}(u)} \big] \tag{5.13}$$

where $>$ denotes the order relation on S^1 which is obtained by transport of the standard order relation on $[0, 1)$ with the mapping i_{t_0} (and which therefore depends on the choice of $t_0 \in S^1$).

For simplicity, let us assume that $\mathbf{g}$ was chosen such that, when restricted onto a suitable open neighborhood V of $\bigcup_j \operatorname{arc}(l^j_{\Sigma})$, it coincides with the restriction of the standard Riemannian metric on $U \cong \mathbb{R}^2$ onto V. Then we have

$$\ll T_1 l'_{\Sigma}(s) \delta^\epsilon_{\Sigma}(\cdot - l_{\Sigma}(s)), \star (T_1 \tilde{l}'_{\Sigma}(u) \delta^\epsilon_{\Sigma}(\cdot - \tilde{l}_{\Sigma}(u))) \gg_{\mathcal{H}_\Sigma}$$

$$= -\operatorname{Tr}(T_1 T_1) \langle l'_{\Sigma}(s), \star \tilde{l}'_{\Sigma}(u) \rangle_{\mathbb{R}^2} \int_\Sigma \delta^\epsilon_{\Sigma}(\sigma - l_{\Sigma}(s)) \delta^\epsilon_{\Sigma}(\sigma - \tilde{l}_{\Sigma}(u)) d\mu_{\mathbf{g}}(\sigma)$$

$$= (l'_{\Sigma}(s)_1 \tilde{l}'_{\Sigma}(u)_2 - l'_{\Sigma}(s)_2 \tilde{l}'_{\Sigma}(u)_1) \int_\Sigma \delta^\epsilon_{\Sigma}(\sigma - l_{\Sigma}(s)) \delta^\epsilon_{\Sigma}(\sigma - \tilde{l}_{\Sigma}(u)) d\mu_{\mathbf{g}}(\sigma) \tag{5.14}$$

Here the last step follows because the Hodge operator $\star : \mathbb{R}^2 \to \mathbb{R}^2$ appearing above is just given by $\star (x_1, x_2) = (x_2, -x_1)$. One can show (cf. [23, 21]) that for every smooth function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ one has

$$\lim_{\epsilon \to 0} \int_0^1 \int_0^1 \Big[f(s, u) \int_\Sigma \delta^\epsilon_{\Sigma}(\sigma - l_{\Sigma}(s)) \delta^\epsilon_{\Sigma}(\sigma - \tilde{l}_{\Sigma}(u)) d\mu_{\mathbf{g}}(\sigma) \Big] ds \ du$$

$$= \sum_{\bar{s}, \bar{u} \text{ with } l_{\Sigma}(\bar{s}) = \tilde{l}_{\Sigma}(\bar{u})} f(\bar{s}, \bar{u}) \frac{1}{|l'_{\Sigma}(\bar{s})_1 \tilde{l}'_{\Sigma}(\bar{u})_2 - l'_{\Sigma}(\bar{s})_2 \tilde{l}'_{\Sigma}(\bar{u})_1|} \tag{5.15}$$

Combining Eqs. (5.12)–(5.15) we obtain

$$\exp(-\tfrac{1}{2} T(l, \tilde{l})) = \exp(\tfrac{\pi i}{k} \operatorname{LK}^*(l, \tilde{l})) \tag{5.16}$$

where we have set

$$\operatorname{LK}^*(l, \tilde{l}) := \frac{1}{2} \sum_{\bar{s}, \bar{u} \text{ with } l_{\Sigma}(\bar{s}) = \tilde{l}_{\Sigma}(\bar{u})} \epsilon(\bar{s}, \bar{u}) \tag{5.17}$$

with

$$\epsilon(\bar{s}, \bar{u}) := \big[1_{l_{S^1}(\bar{s}) < \tilde{l}_{S^1}(\bar{u})} - 1_{l_{S^1}(\bar{s}) > \tilde{l}_{S^1}(\bar{u})} \big] \operatorname{sgn}(l'_{\Sigma}(\bar{s})_1 \tilde{l}'_{\Sigma}(\bar{u})_2 - l'_{\Sigma}(\bar{s})_2 \tilde{l}'_{\Sigma}(\bar{u})_1) \in \{-1, 1\}$$

Clearly, $\operatorname{LK}^*(l, \tilde{l})$ depends on the choice of the point $t_0 \in S^1$. Anyhow, it is closely related to the linking number $\operatorname{LK}(l, \tilde{l})$ of l and $\tilde{l}$ (which does, of course, not depend on t_0). The precise relationship will be given in Proposition 6.1 below.

Until now we have only studied the expression $T(l_j, l_k)$ in the case $j \neq k$. The reason why we have excluded the case $j = k$ so far is that in a naive treatment of the case $j = k$ the so-called "self-linking problem" would appear. One can avoid the "self-linking problem" by introducing an additional regularization procedure which is called "framing". By a "framing" of the link $L = (l_1, l_2, \ldots, l_n)$ we will understand in the sequel (cf. Remark 5.2) a family $(\phi_s)_{s>0}$ of diffeomorphisms of M such that $\phi_s \to \mathrm{id}_M$ uniformly on M (or, at least uniformly on $\bigcup_i \mathrm{arc}(l_i)$) as $s \to 0$. We will call a framing $(\phi_s)_{s>0}$ "admissible" iff it has the following properties:

(F1) Each ϕ_s preserves the orientation of M and also the volume of M (if M is equipped with the Riemannian metric $\mathbf{g}_M$ induced by $\mathbf{g}$)

(F2) Each ϕ_s is "compatible with the torus gauge" in the sense that $\phi_s^*(\mathcal{A}^\perp) = \mathcal{A}^\perp$

(F3) Each two-component link $(l_j, \phi_s \circ l_j)$, $j \leq n$, is admissible for all sufficiently small $s > 0$.

From condition (F2) it follows that each ϕ_s induces a diffeomorphism $\bar{\phi}_s : \Sigma \to \Sigma$ and a linear isomorphism $(\phi_s)_* : \mathcal{A}^\perp \to \mathcal{A}^\perp$ in a natural way (cf. Sec. 9.3 in [23]. Note that $(\phi_s)_*$ does not coincide with $(\phi_s^{-1})^*$).

Remark 5.2. Normally, by a "framing" of a link $L = (l_1, l_2, \ldots, l_n)$ one understands a family $(X_1, X_2, \ldots, X_n)$ where each X_i is a smooth normal vector field on $\mathrm{arc}(l_i)$, i.e. X_i is a mapping $\mathrm{arc}(l_i) \to TM$ such that $X_i(l_i(s)) \in T_{l_i(s)}M$, $s \in [0, 1]$, is normal (w.r.t. to $\mathbf{g}_M$) to the tangent vector $l_i'(s) \in T_{l_i(s)}M$. One can always find a global vector field X on M such that $X_{|\mathrm{arc}(l_i)} = X_i$. As M is compact, X induces a global flow $(\phi_s)_{s\in\mathbb{R}}$ on M. Clearly, $\phi_s \to \mathrm{id}_M$ as $s \to 0$ so X induces a "framing" in the above sense.

With the help of the framing $(\phi_s)_{s>0}$ we can now solve the self-linking problem. The simplest[14] way to do this is the following: We introduce for each ϕ_s a "deformed" versions $\Phi_{B,\phi_s}^\perp$ of $\Phi_B^\perp$. $\Phi_{B,\phi_s}^\perp$ is the unique continuous linear functional $(\mathcal{N}) \to \mathbb{C}$ such that

$$\Phi_{B,\phi_s}^\perp(\exp(i(\cdot, j))) = \exp(i \ll j, m(B) \gg_{\mathcal{H}^\perp}) \exp(-\tfrac{1}{2} \ll (\phi_s)_*(j), C(B)j \gg_{\mathcal{H}^\perp}) \qquad (5.18)$$

for every $j \in \mathcal{N} = \mathcal{A}^\perp$ where $(\phi_s)_* : \mathcal{A}^\perp \to \mathcal{A}^\perp$ is the linear isomorphism mentioned above. We then obtain a "framed" version of $\mathrm{WLO}(L; A_c^\perp, B)$ by setting

$$\mathrm{WLO}(L, \phi_s; A_c^\perp, B) := \lim_{\epsilon \to 0} \Phi_{B,\phi_s}^\perp(\prod_i \mathrm{Tr}(\mathcal{P}\exp(\int_{l_i^\epsilon} (\cdot) + A_c^\perp + Bdt)) \qquad (5.19)$$

Carrying out similar computations as above (for details, see [23, 21]) we then obtain

$$\lim_{\epsilon \to 0} \Phi_{B,\phi_s}^\perp(\prod_i \exp(T_1(\cdot, T_1 \int_0^1 (l_\Sigma^i)'(s)\delta^\epsilon(\cdot - l_i(s))ds)))$$

$$= \left(\prod_j \exp(\int_0^1 l_\mathbb{R}^j(t)\tfrac{d}{dt}B(l_\Sigma^j(t))dt)\right)\left(\prod_{j,k} \exp(\pi i\lambda\, \mathrm{LK}^*(l_j, \phi_s \circ l_k))\right) \quad (5.20)$$

if s is sufficiently small. From Eqs. (5.3), (5.19), (5.20), and (5.10) above we finally obtain (taking into account that for $j \neq k$ one has $\mathrm{LK}^*(l_j, \phi_s \circ l_k) = \mathrm{LK}^*(l_j, l_k)$ if s is sufficiently small)

$$\mathrm{WLO}(L, \phi_s; A_c^\perp, B) = \left(\prod_j \exp(\lambda\pi i\, \mathrm{LK}^*(l_j, \phi_s \circ l_j))\right)\left(\prod_{j\neq k} \exp(\lambda\pi i\, \mathrm{LK}^*(l_j, l_k))\right)$$

$$\times \left(\prod_j \exp(\int_{l_\Sigma^j} A_c^\perp)\right) \exp\left(\sum_{m\in\mathcal{M}(t_0)} \epsilon_m B(\sigma_m)\right) \quad (5.21)$$

---

[14]The standard way of dealing with the self-linking problem consists in replacing some of the loops l_i that appear in the singular terms by their "deformations" $\phi_s \circ l_i$ where s is chosen small enough. If one proceeds in this way one has to deal with each singular term separately. Moreover, the replacement of l_i by $\phi_s \circ l_i$ has to be made "by hand" in the middle of the computations rather the before beginning the computations. Clearly, this is not very elegant.

if $s > 0$ is sufficiently small. Here we have set

$$\mathcal{M}(t_0) := \bigcup_{j=1}^{n} \mathcal{M}_j(t_0), \quad \text{with } \mathcal{M}_j(t_0) := \{(j, u) \mid u \in I_j(t_0)\} \quad \text{for } j \leq n \tag{5.22}$$

and

$$\sigma_m := l_{\Sigma}^{j_m}(u_m), \quad \epsilon_m := \text{sgn}(l_{S^1}^{j_m}; u_m) \quad \text{for } m \in \mathcal{M}(t_0) \tag{5.23}$$

where j_m and u_m are given by $m = (j_m, u_m)$.

Of course, Eq. (5.21) can also derived in the general case, i.e. in the case when assumption (S) above, which we have made in order to simplify the notation, is not fulfilled.

5.2. Non-Abelian G and L has no double points.

Let us now consider the case where G is a Non-Abelian (simple and simply-connected compact) Lie group. In the present subsection we will only consider the special situation where $DP(L) = \emptyset$, i.e. where the link $L = ((l_1, l_2, \ldots, l_n), (\rho_1, \rho_2, \ldots, \rho_n))$ has no double points. For simplicity we will give a detailed computation only for the group $G = SU(N)$ and we will assume that the "colors" $(\rho_1, \rho_2, \ldots, \rho_n)$ of the link L all coincide with the fundamental representation $\rho_{SU(N)}$ of $G = SU(N)$ (see Remark 5.4 below for the case of general G and general link colors).

Let us fix an admissible framing $(\phi_s)_{s>0}$ of L with the following two extra properties:

(H1) For all $j \leq n$ and all sufficiently small $s > 0$ the set of "twist framing double points" of $(l_j, \phi_s \circ l_j)$ (in the sense of Remark 5.3 (1) below) is empty.

(H2) For every $\sigma \in \text{arc}(l_{\Sigma}^j)$ which is not an l_j-self-crossing double point[15] of $(l_j, \phi_s \circ l_j)$ (in the sense of Remark 5.3 (1) below) the points $\bar{\phi}_s(\sigma)$ and $\bar{\phi}_s^{-1}(\sigma)$ lie in different connected components of $\Sigma \backslash \text{arc}(l_{\Sigma}^j)$ provided that $s > 0$ is sufficiently small.

Such a framing will be called "horizontal".

Remark 5.3.   i) In order to explain what the two notions "twist framing double points" and "self-crossing double point" which we have used above mean we first note that if $(l, \tilde{l})$ is an admissible link in $\Sigma \times S^1$ and $p = \pi_{\Sigma}(x) = \pi_{\Sigma}(y)$ where $x \in \text{arc}(l)$, $y \in \text{arc}(\tilde{l})$ then if $\tilde{l}$ is "close" to l normally also y will be "close" to x. But there is one exception: If p is "close" to a double point of l, y need not be "close" to x. In the first case we call p a "twist double point" of $(l, \tilde{l})$ and in the second case a "l-self-crossing double point" (this distinction can be made precise in a very similar way as in Def. 16 in [22]).

ii) As a motivation for the use of the term "horizontal" we remark that if an admissible framing $(\phi_s)_{s>0}$ is induced by a tuple of vector fields $(X_1, X_2, \ldots, X_n)$ like in Remark 5.2 above then for $(\phi_s)_{s>0}$ to be horizontal it is sufficient that each vector field X_j is "horizontal" in the sense that $dt(X_j) = 0$.

We would like to emphasize that here we do not follow the terminology of [22] where the $\mathbb{R}^3$-analogue of this type of framing was not called "horizontal" but "strictly vertical".

Using the Piccard-Lindeloef series expansion we obtain

$$\mathcal{P} \exp\left(\int_{l_j} \hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\text{h}) + Bdt\right)$$

$$= \sum_{m=0}^{\infty} \int_{\triangle_m} \left[D_{u_1}^{l_j}(\hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\text{h}) + Bdt) \cdots D_{u_m}^{l_j}(\hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\text{h}) + Bdt)\right] du \tag{5.24}$$

where we have set $\triangle_m := \{u \in [0, 1]^m \mid u_1 \geq u_2 \geq \cdots \geq u_m\}$ and

$$D_u^{l_j}(\hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\text{h}) + Bdt) := (\hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\text{h}) + Bdt)(l_j'(u))$$

This holds if $\hat{A}^{\perp}$, $A_c^{\perp}$, and B are smooth. In order to be able to work also with general $\hat{A}^{\perp} \in \mathcal{N}^*$ we now use again loop smearing. As in Subsec. 5.1 we will assume again for simplicity that condition (S) above is fulfilled so that we can use the notation $\delta^{\epsilon}(\cdot - l_j(u))$ and make the

---

[15]in which case $\bar{\phi}_s(\sigma) \in \text{arc}(l_{\Sigma}^j)$ would follow

identification $\mathcal{A}_{U \times S^1}^{\perp} \cong C^{\infty}(U \times S^1, \mathbb{R}^2 \otimes \mathfrak{g})$. Recall that the orthogonal-basis $(T_a)_{a \leq \dim(G)}$ of $\mathfrak{g}$ was chosen such that $T_a \in \mathfrak{t}$ for all $a \leq r = \mathrm{rank}(G)$.

Let us now replace all terms of the form $\hat{A}^{\perp}(l'(u))$ by $\sum_a T_a(\hat{A}^{\perp}, T_a(l_{\Sigma})'(u)\delta^{\epsilon}(\cdot - l(u)))$ (here $(\cdot, \cdot)$ denotes again the canonical pairing $\mathcal{N}^* \times \mathcal{N} \to \mathbb{R}$). In particular, we replace $D_u^{l_j}(\hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt)$ by

$$D_u^{l_j^{\epsilon}}(\hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt) :=$$
$$\sum_a T_a(\hat{A}^{\perp}, T_a(l_{\Sigma}^j)'(u)\delta^{\epsilon}(\cdot - l_j(u))) + (A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt)(l_j'(u))$$

where $\epsilon > 0$ and we replace $\mathcal{P} \exp(\int_{l_j} \hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt)$ by

$$\mathcal{P} \exp(\int_{l_j^{\epsilon}} \hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt)$$
$$:= \sum_{m=0}^{\infty} \int_{\triangle_m} \left[D_{u_1}^{l_j^{\epsilon}}(\hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt) \cdots D_{u_m}^{l_j^{\epsilon}}(\hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt) \right] du \quad (5.25)$$

for $\hat{A}^{\perp} \in \mathcal{N}^*$, $A_c^{\perp} \in \mathcal{A}_c^{\perp}$, $\mathrm{h} \in [\Sigma, G/T]$, $B \in \mathcal{B}$. We then have

$$\prod_{j=1}^{n} \mathrm{Tr}_{\rho_j}(\mathcal{P} \exp(\int_{l_j^{\epsilon}} \hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt))$$
$$= \prod_{j=1}^{n} \mathrm{Tr}_{\rho_j} \left[\sum_{m_j=0}^{\infty} \int_{\triangle_{m_j}} \prod_{i_j=1}^{m_j} \left[D_{u_{i_j}}^{l_j^{\epsilon}}(\hat{A}^{\perp} + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt) \right] du \right] \quad (5.26)$$

We will now apply the functional $\Phi_{B,\phi_s}^{\perp}$ on both sides of the previous equation. From the assumption that $DP(L) = \emptyset$ and that the framing $(\phi_s)_{s>0}$ is horizontal it follows for all sufficiently small $s > 0$ that the following statement is true: For all sufficiently small $\epsilon > 0$ the functions $\psi_1, \ldots, \psi_n$ on $\mathcal{N}^*$ given by

$$\psi_j := \mathrm{Tr}_{\rho_j}(\mathcal{P} \exp(\int_{l_j^{\epsilon}} (\cdot) + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt))$$

are "independent" w.r.t. the $\Phi_{B,\phi_s}^{\perp}$ in the sense that

$$\Phi_{B,\phi_s}^{\perp}(\prod_j \psi_j) = \prod_j \Phi_{B,\phi_s}^{\perp}(\psi_j) \quad (5.27)$$

holds (cf. Appendix A.). Thus we can interchange $\Phi_{B,\phi_s}^{\perp}$ with $\prod_{j=1}^{n}$ in Eq. (5.26). We have assumed above that each representation ρ_j equals the fundamental representation $\rho_{SU(N)}$ of $G = SU(N)$. Thus, each Tr_{ρ_j} can be replaced by $\mathrm{Tr}(\cdot) := \mathrm{Tr}_{\mathrm{Mat}(N,\mathbb{C})}(\cdot)$. Clearly, $\Phi_{B,\phi_s}^{\perp}$ "commutes"[16] with $\mathrm{Tr}(\cdot)$ and so we can interchange $\Phi_{B,\phi_s}^{\perp}$ and $\mathrm{Tr}(\cdot)$. By interchanging $\Phi_{B,\phi_s}^{\perp}$ also with $\sum_{m_j}$, $\int_{\triangle_{m_j}} du$, and $\prod_{i=1}^{m_j}$, which can be justified rigorously, we obtain

$$\Phi_{B,\phi_s}^{\perp}\left(\prod_j \mathrm{Tr}(\mathcal{P} \exp(\int_{l_j^{\epsilon}} (\cdot) + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt)) \right)$$
$$= \prod_j \mathrm{Tr} \left[\sum_{m_j} \int_{\triangle_{m_j}} \prod_{i=1}^{m_j} \Phi_{B,\phi_s}^{\perp}\left(D_{u_i}^{l_j^{\epsilon}}(\cdot + A_c^{\perp} + A_{sing}^{\perp}(\mathrm{h}) + Bdt) \right) du \right] \quad (5.28)$$

---

[16]More precisely, we have $\Phi_{B,\phi_s}^{\perp} \circ \mathrm{Tr}(\cdot) = \mathrm{Tr}(\cdot) \circ \Phi_{B,\phi_s}^{\perp}$ where $\Phi_{B,\phi_s}^{\perp} : (\mathcal{N}) \otimes \mathrm{Mat}(N,\mathbb{C}) \to \mathrm{Mat}(N,\mathbb{C})$ and $\mathrm{Tr} : \mathrm{Mat}(N,\mathbb{C}) \to \mathbb{C}$ on the r.h.s. and $\Phi_{B,\phi_s}^{\perp} : (\mathcal{N}) \to \mathbb{C}$ and $\mathrm{Tr} : (\mathcal{N}) \otimes \mathrm{Mat}(N,\mathbb{C}) \to (\mathcal{N})$ on the l.h.s.

Thus, by applying $\lim_{\epsilon \to 0}$ on both sides of the previous equation and interchanging the $\lim_{\epsilon \to 0}$-limit with $\sum_{m_j}$ and $\int_{\triangle_{m_j}} \cdots du$ (this can be justified in a similar way as the analogous steps in the proof of Theorem 4 in [22]) and using Eq. (5.1) we obtain for sufficiently small $s > 0$

$$\text{WLO}(L, \phi_s; A_c^\perp, A_{sing}^\perp(\text{h}), B)$$

$$= \lim_{\epsilon \to 0} \Phi_{B,\phi_s}^\perp \left(\prod_j \text{Tr}(\mathcal{P} \exp(\int_{l_j^\epsilon} (\cdot) + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt)) \right)$$

$$= \prod_j \text{Tr} \left[\sum_{m_j} \int_{\triangle_{m_j}} \prod_{i=1}^{m_j} \lim_{\epsilon \to 0} \Phi_{B,\phi_s}^\perp \left(D_{u_i}^{l_j^\epsilon} (\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt) \right) du \right]$$

$$\overset{(*)}{=} \prod_j \text{Tr} \left[\exp \left(\int_0^1 du \left[\lim_{\epsilon \to 0} \Phi_{B,\phi_s}^\perp \left(D_u^{l_j^\epsilon} (\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt) \right) \right] \right) \right] \quad (5.29)$$

In step $(*)$ we have taken into account that $\lim_{\epsilon \to 0} \Phi_{B,\phi_s}^\perp \left(D_{u_i}^{l_j^\epsilon} (\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt) \right) \in \mathfrak{t}$ (cf. Eq. (5.30) below) so all the factors in the $\prod_{i=1}^{m_j} \cdots$ product in the last but one line in Eq. (5.29) commute with each other and the Piccard-Lindeloef series is reduces to the exponential expression in the last line of Eq. (5.29).

Let us set $l_{\mathbb{R}}^j := i_{t_0}^{-1} \circ l_{S^1}^j - 1/2$, $j \leq n$. Then we have for fixed $s > 0$ and $u \in [0, 1]$

$$\lim_{\epsilon \to 0} \Phi_{B,\phi_s}^\perp \left(D_u^{l_j^\epsilon} (\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt) \right)$$

$$\overset{(*)}{=} \lim_{\epsilon \to 0} \sum_{a=1}^r T_a \ll m(B), T_a(l_\Sigma^j)'(u) \delta^\epsilon (\cdot - l_j(u)) \gg_{\mathcal{H}^\perp} + (A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt)(l_j'(u))$$

$$\overset{(+)}{=} \{ l_{\mathbb{R}}^j(u) \tfrac{d}{du} B(l_\Sigma^j(u)) + B(l_\Sigma^j(u)) \cdot (l_{\mathbb{R}}^j)'(u) \} + (A_c^\perp + A_{sing}^\perp(\text{h}))(l_\Sigma'(u)) \quad (5.30)$$

Here step $(*)$ follows from Eq. (4.2) and step $(+)$ follows in a similar way as Eq. (5.7) above. From Eqs. (5.29) and (5.30) we now obtain (taking into account Eq. (5.10) above)

$$\text{WLO}(L, \phi_s; A_c^\perp, A_{sing}^\perp(\text{h}), B)$$

$$= \prod_{j=1}^n \text{Tr} \left[\exp(\int_{l_\Sigma^j} A_c^\perp) \exp(\int_{l_\Sigma^j} A_{sing}^\perp(\text{h})) \exp(\sum_{m \in \mathcal{M}_j(t_0)} \epsilon_m B(\sigma_m)) \right] \quad (5.31)$$

where $\mathcal{M}_j(t_0)$ and ϵ_m, σ_m for $m \in \mathcal{M}_j(t_0)$ are defined as at the end of Subsec. 5.1.

Remark 5.4. Most of the steps in the computation above can be generalized immediately to the case where G is an arbitrary Non-Abelian (simple and simply-connected compact) Lie group and each ρ_i, $i \leq n$, an arbitrary finite-dimensional representation of G. The only exception is the step where we interchange $\Phi_{B,\phi_s}^\perp$ and Tr. Tr enters the computation because it is the obvious extension of $\text{Tr}_{\rho_{SU(N)}} : G \to \mathbb{C}$ to a linear mapping $\text{Mat}(N, \mathbb{C}) \to \mathbb{C}$. By contrast, $\text{Tr}_{\rho_i} : G \to \mathbb{C}$ will in general not admit an extension to a linear mapping $\text{Mat}(N, \mathbb{C}) \to \mathbb{C}$. Luckily, this complication can be circumvented by taking into account that for every finite-dimensional representation $\rho : G \to \text{GL}(V)$ (and, in particular, for $\rho = \rho_1, \rho_2, \ldots, \rho_n$) one has

$$\text{Tr}_\rho(\mathcal{P} \exp(\int_l A)) = \text{Tr}_{\text{End}(V)}(\rho(\mathcal{P} \exp(\int_l A))) = \text{Tr}_{\text{End}(V)}(\mathcal{P} \exp(\int_l \rho_*(A))) \quad (5.32)$$

where $\rho_* : \mathfrak{g} \to \text{gl}(V)$ is the derived representation of ρ. Clearly, the mappings $\Phi_{B,\phi_s}^\perp$ and $\text{Tr}_{\text{End}(V)}$ can be "interchanged"[17]. The generalization of Eq. (5.31) at which one then arrives is

---

[17] more precisely, we can make use of the equality $\Phi_{B,\phi_s}^\perp \circ \text{Tr}_{\text{End}(V)} = \text{Tr}_{\text{End}(V)} \circ \Phi_{B,\phi_s}^\perp$ where $\Phi_{B,\phi_s}^\perp : (\mathcal{N}) \otimes \text{End}(V) \to \text{End}(V)$ and $\text{Tr} : \text{End}(V) \to \mathbb{C}$ on the r.h.s. and $\Phi_{B,\phi_s}^\perp : (\mathcal{N}) \to \mathbb{C}$ and $\text{Tr} : (\mathcal{N}) \otimes \text{End}(V) \to (\mathcal{N})$ on the l.h.s. are the obvious mappings

the obvious one, i.e. in the general case one simply has

$$\text{WLO}(L, \phi_s; A_c^\perp, A_{sing}^\perp(\text{h}), B)$$
$$= \prod_{j=1}^n \text{Tr}_{\rho_j} \left[\exp(\int_{l_\Sigma^j} A_c^\perp) \exp(\int_{l_\Sigma^j} A_{sing}^\perp(\text{h})) \exp(\sum_{m \in \mathcal{M}_j(t_0)} \epsilon_m B(\sigma_m)) \right] \quad (5.33)$$

5.3. **Non-Abelian G and general L.** As in Subsec. 5.2 let G be a Non-Abelian (simple and simply-connected compact) Lie group. We will now consider the case where L is a general link in Σ. (In order to simplify the notation we will again restrict ourselves to the case where $G = SU(N)$ and where the "colors" $(\rho_1, \rho_2, \ldots, \rho_n)$ of the link L all coincide with the fundamental representation $\rho_{SU(N)}$ of $G = SU(N)$). We will briefly sketch a strategy for evaluating (5.1) which is similar to the strategy used in Sect. 6 in [22]. We first "cut" the loops of L into finitely many subcurves in such a way that the following relations are fulfilled for every $c \in \mathcal{C}(L)$ where $\mathcal{C}(L)$ denotes the set of curves which are obtained by cutting the loops in L (it is not difficult to see that this is always possible if L is admissible)

- $DP(c) = \emptyset$ and $\pi_\Sigma(x) \notin DP(L)$ if $x \in \Sigma \times S^1$ is an endpoint of c.
- There is at most one $c' \in \mathcal{C}(L)$, $c \neq c'$, such that $DP_o(c, c') \neq \emptyset$ and if there is such a c' then $\#DP(c, c') = 1$.

where $DP_o(c, c') := DP(c, c') \setminus \{\pi_\Sigma(x) \mid x \in \Sigma \times S^1 \text{ is an endpoint of } c \text{ or } c'\}$. A "1-cluster" of L is a set of the form $\{c\}$, $c \in \mathcal{C}(L)$, such that $DP_o(c, c') = \emptyset$ for all $c' \in \mathcal{C}(L)$ with $c' \neq c$. A "2-cluster" of L is a set of the form $\{c, c'\}$, $c, c' \in \mathcal{C}(L)$, $c \neq c'$, such that $DP_o(c, c') \neq \emptyset$.

The set of 1-clusters (resp. 2-clusters) of L will be denoted by $Cl_1(L)$ (resp. $Cl_2(L)$). From the properties of $\mathcal{C}(L)$ above it immediately follows that the set $Cl(L)$ defined by $Cl(L) := Cl_1(L) \cup Cl_2(L)$ is a partition of $\mathcal{C}(L)$. If $cl = \{c_1, c_2\} \in Cl_2(L)$ we write $c_1 < c_2$ iff the pair $(\hat{c}_2, \hat{c}_1)$ is positively oriented where $\hat{c}_i$, $i \in \{1, 2\}$, denotes the tangent vector of $\pi_\Sigma \circ c_i$ in the unique double point p of (c_1, c_2).

Let $\epsilon > 0$, $A_c^\perp \in \mathcal{A}_c^\perp$, $\text{h} \in [\Sigma, G/T]$, $B \in \mathcal{B}$ be fixed and let $cl \in Cl(L)$. We set

$$P^{cl^\epsilon}(\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt) := \otimes_{i=1}^{\#cl} \mathcal{P} \exp(\int_{c_i^\epsilon} (\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt)) \quad (5.34)$$

where we have set $\#cl := 1$ (resp. $\#cl := 2$) if $cl \in Cl_1(L)$ (resp. $cl \in Cl_2(L)$) and where c_1 is given (resp. c_1, c_2 are given) by $cl = \{c_1\}$ (resp. $cl = \{c_1, c_2\}$ where $c_1 < c_2$).

It is not difficult to see that there is a linear form β_L on $\otimes_{cl \in Cl(L)}(\otimes^{\#cl} \text{Mat}(N, \mathbb{C}))$ such that for all $\epsilon > 0$ we have $\prod_i \text{Tr}(\mathcal{P} \exp(\int_{l_i^\epsilon} \cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt) = \beta_L \circ (\otimes_{cl \in Cl(L)} P^{cl^\epsilon}(\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt))$. If $s > 0$ is chosen small enough we have

$$\Phi_{B,\phi_s}^\perp \left((\otimes_{cl \in Cl(L)} P^{cl^\epsilon}(\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt)) \right)$$
$$= \otimes_{cl \in Cl(L)} \Phi_{B,\phi_s}^\perp \left(P^{cl^\epsilon}(\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt) \right) \quad (5.35)$$

for all sufficiently small $\epsilon > 0$. This follows in a similar way as Eq. (5.27) above (cf. also Eq. 6.3 in [22]). Eq. (5.35) implies

$$\Phi_{B,\phi_s}^\perp \left(\prod_i \text{Tr}(\mathcal{P} \exp(\int_{l_i^\epsilon} \cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt) \right)$$
$$= \beta_L \left(\otimes_{cl \in Cl(L)} \Phi_{B,\phi_s}^\perp \left(P^{cl^\epsilon}(\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt) \right) \right)$$

for all sufficiently small $s > 0$ and $\epsilon > 0$. In [25] we will show that the limits

$$R^{cl}(\phi_s; A_c^\perp, A_{sing}^\perp(\text{h}), B) := \lim_{\epsilon \to 0} \Phi_{B,\phi_s}^\perp \left(P^{cl^\epsilon}(\cdot + A_c^\perp + A_{sing}^\perp(\text{h}) + Bdt) \right) \quad (5.36)$$

exists. Consequently, we obtain

$$\text{WLF}(L, \phi_s; A_c^\perp, A_{sing}^\perp(\text{h}), B) = \beta_L(\otimes_{cl \in Cl(L)} R^{cl}(\phi_s; A_c^\perp, A_{sing}^\perp(\text{h}), B)) \tag{5.37}$$

The values of $R^{cl}(\phi_s; A_c^\perp, A_{sing}^\perp(\text{h}), B)$ can be computed explicitly using similar techniques as in [22]. In the special case when the framing $(\phi_s)_{s>0}$ is horizontal and $\#cl = 1$ the values of $R^{cl}(\phi_s; A_c^\perp, A_{sing}^\perp(\text{h}), B)$ can be computed in a very similar way as those of the expression $\text{WLO}(L, \phi_s; A_c^\perp, A_{sing}^\perp(\text{h}), B)$ appearing in Eq. (5.29). By contrast, the computation of $R^{cl}(\phi_s; A_c^\perp, A_{sing}^\perp(\text{h}), B)$ for $\#cl = 2$ is rather tedious. We will postpone these computations to [25].

6. The Computation of the WLOs: Step 3

We will now evaluate the whole expression on the right-hand side of Eq. (3.31) in a couple of special cases and then make some remarks concerning the general case.

6.1. **Special case 1: $G = U(1)$ and general L.** Let us go back to the situation of Subsec. 5.1 above, i.e. $G = T = U(1)$ and $\Sigma = S^2$. We want to evaluate the expression

$$\text{WLO}(L, \phi_s) := \int_{\mathcal{A}_c^\perp \times \mathcal{B}} \text{WLO}(L, \phi_s; A_c^\perp, Bdt)$$
$$\times \exp(i\tfrac{k}{2\pi} \ll \star dA_c^\perp, B \gg_{L_t^2(\Sigma, d\mu_\mathbf{g})})(DA_c^\perp \otimes DB) \tag{6.1}$$

where $\mathcal{B} = C^\infty(\Sigma, \mathfrak{t})$ and where $\text{WLO}(L, \phi_s; A_c^\perp, B)$ is as in Eq. (5.19). In order to achieve this we use the identification $\mathcal{A}_c^\perp \cong \mathcal{A}_{\Sigma, \mathfrak{t}}(= \mathcal{A}_\Sigma)$, plug in the right-hand side of Eq. (5.21) into Eq. (6.1) and make use of the Hodge decomposition

$$\mathcal{A}_{\Sigma, \mathfrak{t}} = \mathcal{A}_{ex} \oplus \mathcal{A}_{harm} \oplus \mathcal{A}_{ex}^* \tag{6.2}$$

where $\mathcal{A}_{ex} := \{df \mid f \in C^\infty(\Sigma, \mathfrak{t})\}$, $\mathcal{A}_{ex}^* := \{\star df \mid f \in C^\infty(\Sigma, \mathfrak{t})\}$, and $\mathcal{A}_{harm} := \{A \in \mathcal{A}_{\Sigma, \mathfrak{t}} \mid dA = d(\star A) = 0\}$. As we have assumed here that $\Sigma = S^2$ holds we have $H_\mathbb{R}^1(\Sigma) = 0$ which implies $\mathcal{A}_{harm} = \{0\}$, i.e. Eq. (6.2) reduces to $\mathcal{A}_{\Sigma, \mathfrak{t}} = \mathcal{A}_{ex} \oplus \mathcal{A}_{ex}^*$. Accordingly, we can replace the $\int \cdots DA_c^\perp$ integration in Eq. (6.1) by the integration $\int \int \cdots DA_{ex} DA_{ex}^*$ where DA_{ex}, DA_{ex}^* denote the "Lebesgue measures" on $\mathcal{A}_{ex}$ and $\mathcal{A}_{ex}^*$. Clearly, we have $\int_{l_\Sigma^j} A_{ex} = 0$ and $\ll \star dA_{ex}, B \gg_{L_t^2(\Sigma, d\mu_\mathbf{g})} = 0$ for every $A_{ex} \in \mathcal{A}_{ex}$. This means that the integrand in the modification of Eq. (6.1) just described, does not depend on the variable A_{ex}. Thus the $\int \cdots DA_{ex}$-integration produces just a constant and we obtain

$$\text{WLO}(L, \phi_s) \sim \prod_j \exp(\lambda \pi i \, \text{LK}^*(l_j, \phi_s \circ l_j)) \prod_{j \neq k} \exp(\lambda \pi i \, \text{LK}^*(l_j, l_k))$$
$$\times \int_\mathcal{B} \int_{\mathcal{A}_{ex}^*} \left[\prod_j \exp(\int_{l_\Sigma^j} A_{ex}^*) \right] \left[\prod_{m \in \mathcal{M}(t_0)} \exp(\epsilon_m B(\sigma_m)) \right]$$
$$\times \exp(i\tfrac{k}{2\pi} \ll \star dA_{ex}^*, B \gg_{L_t^2(\Sigma, d\mu_\mathbf{g})}) DA_{ex}^* DB \tag{6.3}$$

Let us assume for a while that l_Σ^j is a Jordan loop in $\Sigma = S^2$. Then there are exactly two connected components K_+ and K_- of $\Sigma \backslash \text{arc}(l_\Sigma^j)$. Here K_+ (resp. K_-) denotes the connected component of $\Sigma \backslash \text{arc}(l_\Sigma^j)$ with the property that the orientation on $\partial K_+ = \partial K_- = \text{arc}(l_\Sigma^j)$ which is induced by that on K_+ (resp. K_-) coincides with (resp. is opposite to) the orientation on $\text{arc}(l_\Sigma^j)$ which is obtained from the standard orientation of S^1 by transport with $l_\Sigma^j : S^1 \to \text{arc}(l_\Sigma^j)$. Stokes' Theorem implies

$$\int_{l_\Sigma^j} A_{ex}^* = \frac{1}{2}(\int_{\partial K_+} A_{ex}^* + \int_{\partial K_-} A_{ex}^*)$$
$$= \frac{1}{2}(\int_{K_+} dA_{ex}^* - \int_{K_-} dA_{ex}^*) = T_1 \ll \star dA_{ex}^*, T_1 \, \text{ind}(l_\Sigma^j; \cdot) \gg_{L_t^2(\Sigma, d\mu_\mathbf{g})}$$

where we have set $\mathrm{ind}(l_\Sigma^j; \cdot) := \frac{1}{2}(1_{K_+} - 1_{K_-})$. This formula can be generalized to the situation where l_Σ^j is not necessarily a Jordan loop but any smooth loop in $\Sigma = S^2$ with the property that $\Sigma \backslash \mathrm{arc}(l_\Sigma^j)$ has only finitely many connected components. In this case we can "decompose" l_Σ^j into finitely many Jordan loops $l_{\Sigma,1}^j, \ldots, l_{\Sigma,m}^j$. More precisely, we can find a finite sequence of (piecewise smooth) Jordan loops $l_{\Sigma,1}^j, \ldots, l_{\Sigma,m}^j$ such that $\mathrm{arc}(l_\Sigma^j) = \bigcup_{i=1}^m \mathrm{arc}(l_{\Sigma,i}^j)$ and $\mathrm{arc}(l_{\Sigma,i}^j) \cap \mathrm{arc}(l_{\Sigma,i'}^j) \subset DP(l_j)$ if $i \neq i'$. Then we have $\int_{l_\Sigma^j} A_{ex}^* = \sum_{i=1}^m \int_{l_{\Sigma,i}^j} A_{ex}^*$. So, setting

$$\mathrm{ind}(l_\Sigma^j; \cdot) := \sum_{i=1}^m \mathrm{ind}(l_{\Sigma,i}^j; \cdot) \tag{6.4}$$

we obtain again

$$\int_{l_\Sigma^j} A_{ex}^* = T_1 \ll \star dA_{ex}^*, T_1 \, \mathrm{ind}(l_\Sigma^j; \cdot) \gg_{L_t^2(\Sigma, d\mu_{\mathbf{g}})} \tag{6.5}$$

Remark 6.1. One can show that for all $\sigma \in \Sigma \backslash \mathrm{arc}(l_\Sigma^j)$ with $\sigma \neq \sigma_0$ we have

$$\mathrm{ind}(l_\Sigma^j; \sigma) - \mathrm{ind}(l_\Sigma^j; \sigma_0) = \mathrm{ind}(l_{\Sigma \backslash \{\sigma_0\}}^j; \sigma) \tag{6.6}$$

where $\mathrm{ind}(l_{\Sigma \backslash \{\sigma_0\}}^j; \sigma)$ denotes the index of the point σ with respect to the loop $l_{\Sigma \backslash \{\sigma_0\}}^j : [0,1] \ni t \mapsto l_\Sigma^j(t) \in \Sigma \backslash \{\sigma_0\} = S^2 \backslash \{\sigma_0\} \cong \mathbb{R}^2$ (or the "winding number" of $l_{\Sigma \backslash \{\sigma_0\}}^j$ around σ). Eq. (6.6) characterizes $\mathrm{ind}(l_\Sigma^j; \cdot)$ on $\Sigma \backslash \mathrm{arc}(l_\Sigma^j)$ completely up to an additive constant. Clearly, this additive constant does not affect the validity of Eq. (6.5). This means that if we had defined $\mathrm{ind}(l_\Sigma^j; \cdot)$ by

$$\mathrm{ind}(l_\Sigma^j; \sigma) := \begin{cases} \mathrm{ind}(l_{\Sigma \backslash \{\sigma_0\}}^j; \sigma) & \text{if } \sigma \neq \sigma_0 \text{ and } \sigma \notin \mathrm{arc}(l_\Sigma^j) \\ 0 & \text{if } \sigma = \sigma_0 \text{ or } \sigma \in \mathrm{arc}(l_\Sigma^j) \end{cases} \tag{6.7}$$

then Eq. (6.5) would still hold. This alternative definition of $\mathrm{ind}(l_\Sigma^j; \cdot)$ (resp. a suitable generalization of it) will be useful in Subsec. 6.3 below.

We will now evaluate the right-hand side of Eq. (6.3) at a heuristic level. In [26] we will sketch how a rigorous treatment can be obtained. Recall that $T_1 = i$. Thus we have

$$\left[\prod_j \exp\left(\int_{l_j} A_{ex}^* \right) \right] \exp\left(i\frac{k}{2\pi} \ll \star dA_{ex}^*, B \gg_{L_t^2(\Sigma, d\mu_{\mathbf{g}})} \right)$$
$$= \exp\left(i\frac{k}{2\pi} \ll \star dA_{ex}^*, B + \frac{2\pi}{k} \sum_j T_1 \, \mathrm{ind}(l_\Sigma^j; \cdot) \gg_{L_t^2(\Sigma, d\mu_{\mathbf{g}})} \right) \tag{6.8}$$

Note that $\ll \star dA_{ex}^*, B + \frac{2\pi}{k} \sum_j T_1 \, \mathrm{ind}(l_\Sigma^j; \cdot) \gg_{L_t^2(\Sigma, d\mu_{\mathbf{g}})}$ vanishes for all $A_{ex}^* \in \mathcal{A}_{ex}^*$ if and only if $B + \frac{2\pi}{k} \sum_j T_1 \, \mathrm{ind}(l_\Sigma^j; \cdot)$ is a constant function, i.e. iff there is a $b \in \mathfrak{t}$ with $B = b - \frac{2\pi}{k} \sum_j T_1 \, \mathrm{ind}(l_\Sigma^j; \cdot)$.

So we obtain, informally,

$$\int_{\mathcal{B}} \int_{\mathcal{A}_{ex}^*} \left[\prod_j \exp(\int_{l_\Sigma^j} A_{ex}^*) \right] \left[\prod_{m \in \mathcal{M}(t_0)} \exp(\epsilon_m B(\sigma_m)) \right]$$

$$\exp(i\tfrac{k}{2\pi} \ll \star dA_{ex}^*, B \gg_{L_\mathfrak{t}^2(\Sigma, d\mu_\mathbf{g})}) DA_{ex}^* DB$$

$$= \int_{\mathcal{B}} \left[\left[\int_{\mathcal{A}_{ex}^*} \exp(i\tfrac{k}{2\pi} \ll \star dA_{ex}^*, B + \tfrac{2\pi}{k} \sum_j T_1 \operatorname{ind}(l_\Sigma^j; \cdot) \gg_{L_\mathfrak{t}^2(\Sigma, d\mu_\mathbf{g})}) DA_{ex}^* \right] \right.$$

$$\left. \times \prod_{m \in \mathcal{M}(t_0)} \exp(\epsilon_m B(\sigma_m)) \right] DB$$

$$= \int_\mathfrak{t} db \left[\int \delta(B - (b - \tfrac{2\pi}{k} \sum_j T_1 \operatorname{ind}(l_\Sigma^j; \cdot))) \prod_{m \in \mathcal{M}(t_0)} \exp(\epsilon_m B(\sigma_m)) DB \right]$$

$$= \left(\prod_{m \in \mathcal{M}(t_0)} \exp(-\epsilon_m 2\pi\lambda \sum_j T_1 \operatorname{ind}(l_\Sigma^j; \sigma_m)) \right) \left(\int_\mathfrak{t} db \prod_{m \in \mathcal{M}(t_0)} \exp(\epsilon_m b) \right) \quad (6.9)$$

Informally, we have

$$\int_\mathfrak{t} \prod_m \exp(\epsilon_m b) \, db = \int_\mathfrak{t} \exp(\sum_m \epsilon_m b) \, db = \delta(\sum_m \epsilon_m) = \delta(\sum_j \operatorname{wind}(l_{S^1}^j)) \quad (6.10)$$

because $\sum_m \epsilon_m = \sum_j \operatorname{wind}(l_{S^1}^j)$ where $\operatorname{wind}(l_{S^1}^j)$ is the winding number of $l_{S^1}^j$.

For evaluating the other factor in Eq. (6.9) we now use the 2-dimensional analogue of the framing procedure of Sec. 5. We replace the expression $\operatorname{ind}(l_\Sigma^j; \sigma_m)$ by $\frac{1}{2}\left[\operatorname{ind}(l_\Sigma^j; \bar{\phi}_s(\sigma_m)) + \operatorname{ind}(l_\Sigma^j; \bar{\phi}_s^{-1}(\sigma_m))\right]$ where $\bar{\phi}_s : \Sigma \to \Sigma$ is as in the paragraph preceding Remark 5.2 above. Taking into account that $T_1 = i$ we then obtain

$$\operatorname{WLO}(L, \phi_s) \sim$$

$$\left(\prod_j \exp(\lambda\pi i \operatorname{LK}^*(l_j, \phi_s \circ l_j)) \right) \left(\prod_{j \neq k} \exp(\lambda\pi i \operatorname{LK}^*(l_j, l_k)) \right) \delta(\sum_j \operatorname{wind}(l_{S^1}^j))$$

$$\times \prod_j \prod_{m \in \mathcal{M}(t_0)} \exp(-\pi i \lambda \epsilon_m \left[\operatorname{ind}(l_\Sigma^j; \bar{\phi}_s(\sigma_m)) + \operatorname{ind}(l_\Sigma^j; \bar{\phi}_s^{-1}(\sigma_m)) \right]) \quad (6.11)$$

We will now make use of the following proposition, which is not difficult to prove.

Proposition 6.1. *If l and $\tilde{l}$ are loops in $\Sigma \times S^1$ which are 0-homologous and which have the additional property that $(l, \tilde{l})$ is admissible in the sense of Subsec. 3.1 then the linking number $\operatorname{LK}(l, \tilde{l})$ of the pair $(l, \tilde{l})$ is well-defined and we have*

$$\operatorname{LK}(l, \tilde{l}) = \operatorname{LK}^*(l, \tilde{l}) - \sum_{u \in I} \epsilon_u \operatorname{ind}(\tilde{l}_\Sigma; \sigma_u) - \sum_{u \in \tilde{I}} \tilde{\epsilon}_u \operatorname{ind}(l_\Sigma; \tilde{\sigma}_u) \quad (6.12)$$

where we have set $\sigma_u := l_\Sigma(u)$, $\epsilon_u := \operatorname{sgn}(l_{S^1}; u)$ for $u \in I := l_{S^1}^{-1}(\{t_0\})$ and $\tilde{\sigma}_u := \tilde{l}_\Sigma(u)$, $\tilde{\epsilon}_u := \operatorname{sgn}(\tilde{l}_{S^1}; u)$ for $u \in \tilde{I} := \tilde{l}_{S^1}^{-1}(\{t_0\})$

From this proposition it follows that for sufficiently small $s > 0$ we have

$$\operatorname{LK}(l_j, \phi_s \circ l_j) = \operatorname{LK}^*(l_j, \phi_s \circ l_j) - \sum_{m \in \mathcal{M}_j(t_0)} \epsilon_m \left[\operatorname{ind}(l_\Sigma^j; \bar{\phi}_s(\sigma_m)) + \operatorname{ind}(l_\Sigma^j; \bar{\phi}_s^{-1}(\sigma_m)) \right] \quad (6.13)$$

and that

$$\sum_{j \neq k} \operatorname{LK}(l_j, l_k) = \sum_{j \neq k} \operatorname{LK}^*(l_j, l_k) - \sum_j \sum_{m \in \mathcal{M}(t_0) \setminus \mathcal{M}_j(t_0)} 2\epsilon_m \operatorname{ind}(l_\Sigma^j; \sigma_m) \quad (6.14)$$

(here we have used that for every $m \in \mathcal{M}(t_0) \backslash \mathcal{M}_j(t_0)$ and every sufficiently small $s > 0$ one has $\mathrm{ind}(l_\Sigma^j; \bar{\phi}_s(\sigma_m)) + \mathrm{ind}(l_\Sigma^j; \bar{\phi}_s^{-1}(\sigma_m)) = 2\,\mathrm{ind}(l_\Sigma^j; \sigma_m))$. So, if every l_j is 0-homologous (in which case $\sum_{j=1}^n \mathrm{wind}(l_{S^1}^j) = \sum_{j=1}^n 0 = 0$ holds) we finally obtain from Eqs. (6.11), (6.13), and (6.14)

$$\mathrm{WLO}(L, \phi_s) \sim \left(\prod_j \exp(\lambda \pi i \, \mathrm{LK}(l_j, \phi_s \circ l_j)) \right) \left(\prod_{j \neq k} \exp(\lambda \pi i \, \mathrm{LK}(l_j, l_k)) \right) \qquad (6.15)$$

for sufficiently small $s > 0$. This is exactly the expression that was obtained by other methods, see, e.g., [3, 33].

Remark 6.2. Eq. (6.12) only holds when we use the original definition of $\mathrm{ind}(l_\Sigma^j; \cdot)$ given in Eq. (6.4). If we had defined $\mathrm{ind}(l_\Sigma^j; \cdot)$ by Eq. (6.7) instead we would have obtained a correction factor of the form $\exp(C \cdot \sum_m \epsilon_m)$ in Eq. (6.15) where C is a suitable constant. Of course, if every loop is 0-homologous we have $\mathrm{wind}(l_{S^1}^j) = 0$, $j \leq n$, and thus also $\sum_m \epsilon_m = \sum_{j=1}^n \mathrm{wind}(l_{S^1}^j) = 0$. So the correction factor is trivial and we obtain again Eq. (6.15)

6.2. Special case 2: $G = SU(2)$ and L consists of vertical loops with arbitrary colors.
Let us now consider the case where $G = SU(N)$ and where Σ is an arbitrary compact oriented surface. (Later we will restrict ourselves to the special case $N = 2$). We will assume in the present subsection that in the colored link $L = ((l_1, l_2, \ldots, l_n), (\rho_1, \rho_2, \ldots, \rho_n))$, which we have fixed in Subsec. 3.1 each l_i, $i \leq n$, is a vertical loop (cf. Subsec. 2.1) above the point σ_i, $i \leq n$. The colors ρ_i can be arbitrary. In this situation we have

$$\prod_i \mathrm{Tr}_{\rho_i}(\mathcal{P}\exp(\int_{l_i} \hat{A}^\perp + A_c^\perp + A_{sing}^\perp(\mathrm{h}) + Bdt)) = \prod_i \mathrm{Tr}_{\rho_i}(\exp(\int_{l_i} Bdt) = \prod_i \mathrm{Tr}_{\rho_i}(\exp(B(\sigma_i)))$$

so we can conclude, informally, that the integral $\int_{\hat{A}^\perp} \prod_i \mathrm{Tr}_{\rho_i}(\mathcal{P}\exp(\int_{l_i} \hat{A}^\perp + A_c^\perp + A_{sing}^\perp(\mathrm{h}) + Bdt))d\hat{\mu}_B^\perp(\hat{A}^\perp)$ appearing in Eq. (3.31) coincides with $\prod_i \mathrm{Tr}_{\rho_i}(\exp(B(\sigma_i)))$. For $G = SU(N)$, for which $c_G = N$, we thus obtain[18] from Eq. (3.31)

$$\mathrm{WLO}(L) \sim \sum_\mathrm{h} \int_\mathcal{B} \left(\prod_j \mathrm{Tr}_{\rho_j}\left[\exp(B(\sigma_j)) \right] \right) \left(\int_{\mathcal{A}_c^\perp} \exp(i\tfrac{k+N}{2\pi} \ll \star dA_c^\perp, B \gg_{L_\mathfrak{t}^2(\Sigma, d\mu_\mathbf{g})}) DA_c^\perp \right)$$
$$\exp(i\tfrac{k+N}{2\pi} \ll \star dA_{sing}^\perp(\mathrm{h}), B \gg) \det{}_{reg}\left(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0}) \right) DB \quad (6.16)$$

Eq. (6.16) can be considered as a reformulation of Eq. (7.24) in [10]. The evaluation of Eq. (6.16) which we will give now differs only slightly from the analogous treatment given in Secs. 7.1–7.6 in [10].

Let us use again the Hodge decomposition (6.2) of $\mathcal{A}_c^\perp \cong \mathcal{A}_{\Sigma, \mathfrak{t}}$. In the present subsection we do not assume that $\Sigma \cong S^2$ holds so the space $\mathcal{A}_{harm} \cong H_\mathbb{R}^1(\Sigma) \otimes \mathfrak{t}$ need not vanish. After replacing the $\int \cdots DA_c^\perp$-integration in Eq. (6.16) by $\int \int \int \cdots DA_{ex}DA_{harm}DA_{ex}^*$, where DA_{ex}, DA_{harm}, DA_{ex}^* denote the "Lebesgue measures" on the obvious spaces, we obtain

$$\int_{\mathcal{A}_c^\perp} \exp(i\tfrac{k+N}{2\pi} \ll \star dA_c^\perp, B \gg_{L_\mathfrak{t}^2(\Sigma, d\mu_\mathbf{g})}) DA_c^\perp \sim \int_{\mathcal{A}_{ex}^*} \exp(i\tfrac{k+N}{2\pi} \ll \star dA_{ex}^*, B \gg_{L_\mathfrak{t}^2(\Sigma, d\mu_\mathbf{g})}) DA_{ex}^*$$

because the $\int \cdots DA_{ex}$- and $\int \cdots DA_{harm}$-integrations are trivial. Taking into account that $\ll \star dA_{ex}^*, B \gg_{L_\mathfrak{t}^2(\Sigma, d\mu_\mathbf{g})}$ vanishes for every A_{ex}^* if and only if $B \in C^\infty(\Sigma, P)$ is constant we

---

[18]note that in contrast to the situation in Subsec. 6.1 and Subsec. 6.2 no framing is necessary in the present subsection. For this reason we will denote the WLOs just by $\mathrm{WLO}(L)$ instead of $\mathrm{WLO}(L; \phi_s)$

obtain

$$\text{WLO}(L) \sim \sum_{\text{h}} \int_{\mathfrak{t}} db \int_{\mathcal{B}} DB \Big[\delta(B-b) \big(\prod_{j} \text{Tr}_{\rho_j} \big[\exp(B(\sigma_j)) \big] \big)$$

$$\times \exp\big(i \tfrac{k+N}{2\pi} \ll \star dA_{sing}^{\perp}(\text{h}), B \gg \big) \det_{reg}\big(1_{\mathfrak{g}_0} - \exp(\text{ad}(B)_{|\mathfrak{g}_0})\big) \Big]$$

$$= \sum_{\text{h}} \int_{P} db \big(\prod_{j} \text{Tr}_{\rho_j} \big[\exp(b) \big] \big) \exp\big(i \tfrac{k+N}{2\pi} \ll \star dA_{sing}^{\perp}(\text{h}), b \gg \big)$$

$$\times \det\big(1_{\mathfrak{g}_0} - \exp(\text{ad}(b)_{|\mathfrak{g}_0})\big)^{\chi(\Sigma)/2}$$

where db is the Lebesgue measure on $\mathfrak{t}$. Here we have used that

$$\det_{reg}\big(1_{\mathfrak{g}_0} - \exp(\text{ad}(B)_{|\mathfrak{g}_0})\big) = \det\big(1_{\mathfrak{g}_0} - \exp(\text{ad}(b)_{|\mathfrak{g}_0})\big)^{\chi(\Sigma)/2}$$

if B equals the constant function b, cf. Eq. (3.30).

From Eq. (3.9) and the definition of $n(\text{h})$ and $A_{sing}^{\perp}(\text{h})$ it follows immediately that

$$\tfrac{k+N}{2\pi} \ll \star dA_{sing}^{\perp}(\text{h}), b \gg = \tfrac{k+N}{2\pi} n(\text{h}) \cdot b \tag{6.17}$$

where "$\cdot$" denotes the scalar product on $\mathfrak{t}$ induced by $(\cdot, \cdot)_{\mathfrak{g}}$. From $\exp(\text{ad}(b)_{|\mathfrak{g}_0}) = \text{Ad}(\exp(b))_{|\mathfrak{g}_0}$ we obtain

$$\text{WLO}(L) \sim \sum_{\text{h}} \int_{P} db \ f(\exp(b)) \exp\big(i \tfrac{k+N}{2\pi} \ n(\text{h}) \cdot b\big) \tag{6.18}$$

where we have set $f(t) := \big(\prod_{j} \text{Tr}_{\rho_j}(t) \big) \det\big(1_{\mathfrak{g}_0} - \text{Ad}(t)_{|\mathfrak{g}_0}\big)^{\chi(\Sigma)/2}$, $t \in T$.

For simplicity, let us restrict ourselves to the the special case $N = 2$, i.e. $G = SU(2)$. We can then choose T to be the maximal torus $\{ \big(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix} \big) \mid \theta \in [0, 2\pi] \}$ and $P \subset \mathfrak{t} = \mathbb{R}\tau = \{\theta \cdot \tau \mid \theta \in \mathbb{R}\}$ to be the (open) alcove

$$P := \{\theta \cdot \tau \mid \theta \in (0, \pi)\} \quad \text{where } \tau := \big(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} \big) \tag{6.19}$$

Taking into account Eq. (2.23) and

$$\{n(\text{h}) \mid \text{h} \in [\Sigma, G/T]\} = \ker(\exp_{|\mathfrak{t}}) = 2\pi\mathbb{Z} \cdot \tau \tag{6.20}$$

$$\tau \cdot \tau = - \text{Tr}(\tau\tau) = 2 \tag{6.21}$$

$$\det\big(1_{\mathfrak{g}_0} - \text{Ad}(\exp(x \cdot \tau)))_{|\mathfrak{g}_0}\big) = \sin(x)^2 \tag{6.22}$$

$$\text{Tr}_{\rho_j}(\exp(x \cdot \tau)) = \frac{\sin(d_j x)}{\sin(x)} \tag{6.23}$$

where d_j is the dimension of the representation ρ_j we obtain, informally,

$$\text{WLO}(L) \sim \sum_{m=-\infty}^{\infty} \int_{(0,\pi)} e^{im(k+2)2x} f(e^{x\tau}) \ dx$$

$$= \int 1_{(0,\pi)}(x) \big(\sum_{m=-\infty}^{\infty} e^{im(k+2)2x} \big) f(e^{x\tau}) \ dx$$

$$\overset{(*)}{=} \int 1_{(0,\pi)}(x) \delta_{\frac{\pi}{k+2}\mathbb{Z}}(x) f(e^{x\tau}) \ dx = \sum_{l=1}^{k+1} f(e^{\frac{\pi}{k+2}l\tau})$$

$$= \sum_{l=1}^{k+1} \prod_{j} \frac{\sin(\frac{ld_j\pi}{k+2})}{\sin(\frac{l\pi}{k+2})} \sin^{2-2g}(\tfrac{l\pi}{k+2}) \tag{6.24}$$

where $\delta_{\frac{\pi}{k+2}\mathbb{Z}}$ is the periodic delta-function associated to the lattice $\frac{\pi}{k+2}\mathbb{Z}$ in $\mathbb{R}$ and where g denotes the genus of Σ. In step $(*)$ we have pretended that we can apply the Poisson summation formula

$\int \phi(x) \big(\sum_{m=-\infty}^{\infty} e^{im(k+2)2x} \big) dx = \int \phi(x) \delta_{\frac{\pi}{k+2}\mathbb{Z}}(x) dx$, which holds, e.g., if ϕ is a smooth function of rapid decrease. Clearly, the function $1_{(0,\pi)}(x) f(e^{x\tau})$ is not smooth, it even has a singularity at the points $x = 0$ and $x = \pi$ if $\chi(\Sigma) < 0$. In a rigorous treatment of the above derivation where, among other things, "loop smearing" is used in a suitable way this complication can probably be avoided.

6.3. Special case 3: $G = SU(2)$ and L has standard colors and no double points.

Let us consider again the case where $G = SU(N)$. (Later we will restrict ourselves to the special case $N = 2$). We will now assume that the colored link $L = ((l_1, l_2, \ldots, l_n), (\rho_1, \rho_2, \ldots, \rho_n))$ which we have fixed in Subsec. 3.1 is admissible and has no double points and that each ρ_j is equal to the fundamental representation $\rho_{SU(N)}$ of $SU(N)$.

As $G = SU(N)$ is simply-connected Σ can be an arbitrary (oriented compact) surface. Note, however, that the case $\Sigma \not\cong S^2$ is slightly more complicated than the case $\Sigma \cong S^2$. Firstly, in the Hodge decomposition (6.2) of $\mathcal{A}_c^{\perp} \cong \mathcal{A}_{\Sigma,\mathfrak{t}}$ the space $\mathcal{A}_{harm}$ is not trivial if $\Sigma \not\cong S^2$. Secondly, in the case $\Sigma \not\cong S^2$ the definition of the functions $\mathrm{ind}(l_{\Sigma}^j; \cdot)$ for general loops l_{Σ}^j in Σ is also more complicated than in the case $\Sigma \cong S^2$.

In order to circumvent these complications in the present paper we will make the additional assumption that for the link L considered each l_{Σ}^j is 0-homotopic. From this and $DP(L) = \emptyset$ it then follows that $\Sigma \backslash \mathrm{arc}(l_{\Sigma}^j)$ will have exactly two connected components and we can then define the functions $\mathrm{ind}(l_{\Sigma}^j; \cdot)$ for arbitrary Σ in a similar way as in Subsec. 6.1 above for the case $\Sigma = S^2$. As in the case $\Sigma = S^2$ there is a certain freedom in defining $\mathrm{ind}(l_{\Sigma}^j; \cdot)$. It turns out that it has several advantages to define $\mathrm{ind}(l_{\Sigma}^j; \cdot)$ in analogy to Remark 6.1, i.e. to fix the additive constant mentioned in Remark 6.1 by demanding that

$$\mathrm{ind}(l_{\Sigma}^j; \sigma_0) = 0 \tag{6.25}$$

holds. As in Subsec. 5.2 let $(\phi_s)_{s>0}$ be a horizontal framing of L. In view of Eq. (3.31) and Eq. (5.1) let us now set

$$\mathrm{WLO}(L; \phi_s) :=$$
$$\sum_{\mathrm{h} \in [\Sigma, G/T]} \int_{\mathcal{A}_c^{\perp} \times \mathcal{B}} \mathrm{WLO}(L, \phi_s; A_c^{\perp}, A_{sing}^{\perp}(\mathrm{h}), B) \exp(i\tfrac{k+c_G}{2\pi} \ll \star dA_{sing}^{\perp}(\mathrm{h}), B \gg)$$
$$\times \det{}_{reg}\big(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\big) \exp(i\tfrac{k+c_G}{2\pi} \ll \star dA_c^{\perp}, B \gg_{L_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathfrak{g}})}) (DA_c^{\perp} \otimes DB)$$

From Eq. (5.31) we obtain (for small $s > 0$)

$$\mathrm{WLO}(L; \phi_s) \sim$$
$$\sum_{\mathrm{h} \in [\Sigma, G/T]} \int_{\mathcal{B}} \Big[\Big[\int_{\mathcal{A}_c^{\perp}} \prod_j \mathrm{Tr} \big[\exp(\sum_{m \in \mathcal{M}_j(t_0)} \epsilon_m B(\sigma_m)) \exp(\int_{l_{\Sigma}^j} A_c^{\perp}) \exp(\int_{l_{\Sigma}^j} A_{sing}^{\perp}(\mathrm{h})) \big]$$
$$\times \exp(i\tfrac{k+N}{2\pi} \ll \star dA_c^{\perp}, B \gg_{L_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathfrak{g}})})) DA_c^{\perp} \Big]$$
$$\times \exp(i\tfrac{k+N}{2\pi} \ll \star dA_{sing}^{\perp}(\mathrm{h}), B \gg)) \det{}_{reg}\big(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0})\big) DB \tag{6.26}$$

Let us again use the Hodge decomposition $\mathcal{A}_c^{\perp} \cong \mathcal{A}_{\Sigma,\mathfrak{t}} = \mathcal{A}_{ex} \oplus \mathcal{A}_{harm} \oplus \mathcal{A}_{ex}^*$ and replace the $\int \cdots DA_c^{\perp}$-integration by $\int \int \int \cdots DA_{ex} DA_{harm} DA_{ex}^*$. Clearly, $\int_{l_{\Sigma}^j} A_{ex} = 0$ and from the assumption that each l_{Σ}^j is 0-homotopic it follows that also $\int_{l_{\Sigma}^j} A_{harm} = 0$ for all $A_{harm} \in \mathcal{A}_{harm}$.

Thus the $\int \cdots DA_{ex}$- and $\int \cdots DA_{harm}$-integrations are trivial and we obtain

$\mathrm{WLO}(L; \phi_s) \sim$

$$\sum_{\mathrm{h}\in[\Sigma, G/T]} \int_{\mathcal{B}} \left\{ \int_{\mathcal{A}_{ex}^*} \prod_j \mathrm{Tr}\left[\exp\left(\sum_{m\in\mathcal{M}_j(t_0)} \epsilon_m B(\sigma_m) \right) \exp\left(\int_{l_\Sigma^j} A_{ex}^* \right) \exp\left(\int_{l_\Sigma^j} A_{sing}^\perp(\mathrm{h}) \right) \right] \right.$$

$$\times \exp\left(i\frac{k+N}{2\pi} \ll \star dA_{ex}^*, B \gg_{L_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathbf{g}})} \right) DA_{ex}^* \Big\}$$

$$\times \exp\left(i\frac{k+N}{2\pi} \ll \star dA_{sing}^\perp(\mathrm{h}), B \gg \right) \det_{reg}\left(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0}) \right) DB \quad (6.27)$$

From a straightforward generalization of Eq. (6.5) we obtain (recall that $T_a \in \mathfrak{t}$ for $a \leq r$)

$$\int_{l_\Sigma^j} A_{ex}^* = \sum_{a=1}^r T_a \ll \star dA_{ex}^*, T_a \,\mathrm{ind}(l_\Sigma^j; \cdot) \gg_{L_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathbf{g}})} \quad (6.28)$$

Taking into account that[19] for $b \in \mathfrak{t}$ we have

$$\mathrm{Tr}(\exp(b)) = \mathrm{Tr}_{\rho_{SU(N)}}(\exp(b)) = \sum_{\alpha\in W_{\rho_{SU(N)}}} \exp(\alpha(b)) \quad (6.29)$$

where $W_{\rho_{SU(N)}}$ is the set of infinitesimal weights $\alpha : \mathfrak{t} \to i\mathbb{R}$ of $\rho_{SU(N)}$ and setting

$$A := W_{\rho_{SU(N)}} \times \cdots \times W_{\rho_{SU(N)}} \quad (6.30)$$

(for an element $(\alpha_1, \ldots, \alpha_n) \in A$ we will often use the shorthand $\underline{\alpha}$) we thus have

$$\prod_j \mathrm{Tr}\left[\exp\left(\sum_{m\in\mathcal{M}_j(t_0)} \epsilon_m B(\sigma_m) \right) \exp\left(\int_{l_\Sigma^j} A_{ex}^* \right) \exp\left(\int_{l_\Sigma^j} A_{sing}^\perp(\mathrm{h}) \right) \right]$$

$$= \sum_{\underline{\alpha}\in A} \prod_{j=1}^n \exp\left(\alpha_j\left(\sum_{m\in\mathcal{M}_j(t_0)} \epsilon_m B(\sigma_m) + \int_{l_\Sigma^j} A_{ex}^* + \int_{l_\Sigma^j} A_{sing}^\perp(\mathrm{h}) \right) \right)$$

$$= \sum_{\underline{\alpha}\in A} \exp\left(\sum_j \sum_{m\in\mathcal{M}_j(t_0)} \epsilon_m \alpha_j(B(\sigma_m)) + \int_{l_\Sigma^j} \alpha_j(A_{sing}^\perp(\mathrm{h})) \right)$$

$$\times \exp\left(-i \ll \star dA_{ex}^*, \mathrm{ind}(L, \underline{\alpha}) \gg_{L_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathbf{g}})} \right) \quad (6.31)$$

where we have set

$$\mathrm{ind}(L, \underline{\alpha}) := -\sum_{j'} \sum_{a\leq r} \frac{1}{i} \alpha_{j'}(T_a) \, T_a \,\mathrm{ind}(l_\Sigma^{j'}; \cdot) \quad (6.32)$$

Note that the function $\frac{1}{i}\alpha_j$ takes values in $\mathbb{R}$. So $\mathrm{ind}(L, \underline{\alpha})$ is a well-defined element of $L_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathbf{g}})$.

We now regularize the expressions $B(\sigma_m)$ using "framing" as in Subsec. 6.1. This amounts to replacing $B(\sigma_m)$ by $\frac{1}{2}\left[B(\bar{\phi}_s(\sigma_m)) + B(\bar{\phi}_s^{-1}(\sigma_m)) \right]$. Then we obtain (for sufficiently small $s > 0$)

$\mathrm{WLO}(L; \phi_s)$

$$\sim \sum_{\mathrm{h}\in[\Sigma, G/T]} \int_{\mathcal{B}} \sum_{\underline{\alpha}\in A} \left\{ \int_{\mathcal{A}_{ex}^*} \exp\left(i \ll \star dA_{ex}^*, \frac{k+N}{2\pi} B - \mathrm{ind}(L, \underline{\alpha}) \gg_{L_{\mathfrak{t}}^2(\Sigma, d\mu_{\mathbf{g}})} \right) DA_{ex}^* \right\}$$

$$\times \exp\left(\sum_j \sum_{m\in\mathcal{M}_j(t_0)} \epsilon_m \frac{1}{2} \alpha_j\left(B(\bar{\phi}_s(\sigma_m)) + B(\bar{\phi}_s^{-1}(\sigma_m)) \right) \right) \det_{reg}\left(1_{\mathfrak{g}_0} - \exp(\mathrm{ad}(B)_{|\mathfrak{g}_0}) \right)$$

$$\times \exp\left(\sum_j \int_{l_\Sigma^j} \alpha_j(A_{sing}^\perp(\mathrm{h})) \right) \exp\left(i\frac{k+N}{2\pi} \ll \star dA_{sing}^\perp(\mathrm{h}), B \gg \right) DB \quad (6.33)$$

---

[19]note that, as $\rho_{SU(N)}$ is the fundamental representation of $SU(N)$, all the weights that appear in the character associated to $\rho_{SU(N)}$ have multiplicity one

Similarly as in Subsec. 6.2 we can argue, informally[20], that $\int_{\mathcal{A}_{ex}^*} \exp(i \ll \star dA_{ex}^*, \frac{k+N}{2\pi}B - \text{ind}(L,\underline{\alpha})) \gg_{L_t^2(\Sigma,d\mu_{\mathbf{g}})}) DA_{ex}^*$ vanishes unless $B - \frac{2\pi}{k+N}\text{ind}(L,\underline{\alpha})$ is a constant function taking[21] values in P. In other words: the aforementioned integral vanishes unless there is a $b \in P$ such that $B = b + \frac{2\pi}{k+N}\text{ind}(L,\underline{\alpha})$ holds. Accordingly, let us replace the $\int \cdots DB$-integration by the integration

$$\int_P db \left[\int_{\mathcal{B}} \cdots \delta(B - b - \tfrac{2\pi}{k+N}\text{ind}(L,\underline{\alpha})) DB \right]$$

Let us set $\epsilon_j := \sum_{m \in \mathcal{M}_j(t_0)} \epsilon_m = \text{wind}(l_{S^1}^j)$ and choose for each j a fixed element of $\{\sigma_m \mid m \in \mathcal{M}_j(t_0)\}$ which we will denote σ_j (if $\mathcal{M}_j(t_0)$ is empty we choose an arbitrary point of $\text{arc}(l_\Sigma^j)$ for σ_j). Moreover, let P^Σ denote the set of all mappings $\Sigma \to P$ and 1_{P^Σ} the corresponding indicator function, i.e. for a function $B : \Sigma \to \mathfrak{t}$ we have

$$1_{P^\Sigma}(B) = \begin{cases} 1 & \text{if Image}(B) \subset P \\ 0 & \text{otherwise} \end{cases}$$

We obtain (for small $s > 0$)

$\text{WLO}(L; \phi_s)$

$$\sim \sum_{\underline{\alpha} \in A} \sum_{\mathbf{h} \in [\Sigma, G/T]} \int_P db \Bigg(1_{P^\Sigma}(B) \exp\Big(\sum_j \epsilon_j \frac{1}{2}\big[\alpha_j\big(B(\bar{\phi}_s(\sigma_j))\big) + \alpha_j\big(B(\bar{\phi}_s^{-1}(\sigma_j))\big)\big]\Big)$$

$$\times \Big[\exp\Big(\sum_j \int_{l_\Sigma^j} \alpha_j(A_{sing}^\perp(\mathbf{h}))\Big) \exp\big(i\tfrac{k+N}{2\pi} \ll \star dA_{sing}^\perp(\mathbf{h}), \tfrac{2\pi}{k+N}\text{ind}(L,\underline{\alpha}) \gg\big) \Big]$$

$$\times \det{}_{reg}\big(1_{\mathfrak{g}_0} - \exp(\text{ad}(B)_{|\mathfrak{g}_0})\big) \exp\big(i\tfrac{k+N}{2\pi} \ll \star dA_{sing}^\perp(\mathbf{h}), b \gg\big) \Bigg)_{|B = b + \frac{2\pi}{k+N}\text{ind}(L,\underline{\alpha})}$$

$$\overset{(*)}{=} \sum_{\underline{\alpha} \in A} \sum_{\mathbf{h} \in [\Sigma, G/T]} \int_P db \Bigg(1_{P^\Sigma}(B) \exp\Big(\sum_j \epsilon_j \frac{1}{2}\big[\alpha_j\big(B(\bar{\phi}_s(\sigma_j))\big) + \alpha_j\big(B(\bar{\phi}_s^{-1}(\sigma_j))\big)\big]\Big)$$

$$\times \Big[1 \Big] \times \det{}_{reg}\big(1_{\mathfrak{g}_0} - \exp(\text{ad}(B)_{|\mathfrak{g}_0})\big) \exp\big(i\tfrac{k+N}{2\pi} n(\mathbf{h}) \cdot b\big) \Bigg)_{|B = b + \frac{2\pi}{k+N}\text{ind}(L,\underline{\alpha})}$$

$$\overset{(**)}{=} \sum_{\underline{\alpha} \in A} \sum_{\mathbf{h}} \int_P db \big(\exp(i\tfrac{k+N}{2\pi} n(\mathbf{h}) \cdot b)\big) \Bigg(1_{P^\Sigma}(B) \exp\Big(\sum_j \epsilon_j \frac{1}{2}\big[\alpha_j\big(B(\bar{\phi}_s(\sigma_j)) + B(\bar{\phi}_s^{-1}(\sigma_j))\big)\big]\Big)$$

$$\times \prod_{t=1}^\mu \det\big(1_{\mathfrak{g}_0} - \text{Ad}(\exp(B(\sigma_{X_t})))_{|\mathfrak{g}_0}\big)^{\chi(X_t)/2} \Bigg)_{|B = b + \frac{2\pi}{k+N}\text{ind}(L,\underline{\alpha})} \tag{6.34}$$

Here step $(*)$ follows from Eq. (6.17) and the relation

$$\sum_{a=1}^r T_a \ll \star dA_{sing}^\perp(\mathbf{h}), T_a \text{ind}(l_\Sigma^j; \cdot) \gg = \int_{l_\Sigma^j} A_{sing}^\perp(\mathbf{h}) \tag{6.35}$$

which is not difficult to show[22]. Step $(**)$ follows from

$$\det{}_{reg}\big(1_{\mathfrak{g}_0} - \exp(\text{ad}(B)_{|\mathfrak{g}_0})\big) = \prod_{t=1}^\mu \det\big(1_{\mathfrak{g}_0} - \text{Ad}(\exp(B(\sigma_{X_t})))_{|\mathfrak{g}_0}\big)^{\chi(X_t)/2} \tag{6.36}$$

---

[20]we expect that it is possible to avoid this heuristic argument and to give a fully rigorous treatment instead, cf. point (4) in Subsec. 7.2

[21]that $B - \frac{2\pi}{k+N}\text{ind}(L,\underline{\alpha})$ must take values in P follows from $\text{ind}(L,\underline{\alpha})(\sigma_0) = 0$, cf. Eq. (6.25)

[22]if we had not defined $\text{ind}(l_\Sigma^j; \cdot)$ such that Eq. (6.25) holds then we would have to replace Eq. (6.35) by the equation $\sum_{a=1}^r T_a \ll \star dA_{sing}^\perp(\mathbf{h}), T_a \text{ind}(l_\Sigma^j; \cdot) \gg = n(\mathbf{h}) \cdot \text{ind}(l_\Sigma^j; \sigma_0) + \int_{l_\Sigma^j} A_{sing}^\perp(\mathbf{h})$

(here we have fixed $\sigma_{X_t} \in X_t$ for each $t \leq \mu$). Eq. (6.36) follows from Eq. (3.30) if we take into account that each function $B = b + \frac{2\pi}{k+N}\text{ind}(L,\underline{\alpha})$ is a "step function" in the sense of Subsec. 3.5.

For simplicity, let us now restrict ourselves to the special case where $N = 2$, i.e. $G = SU(2)$. Then, after informally interchanging the $\sum_{h\in[\Sigma,G/T]}$-summation with the $\int_P$-integration and taking into account Eq. (6.19) and the relation $1_{P\Sigma}(B) = 1_P(B(\sigma_0)) \cdot 1_{P\Sigma}(B) = 1_P(x\tau) \cdot 1_{P\Sigma}(B)$ for $B = x\tau + \frac{2\pi}{k+2}\text{ind}(L,\underline{\alpha})$, which holds because $\text{ind}(L,\underline{\alpha})(\sigma_0) = 0$, we obtain from Eq. (6.34)

$$
\begin{aligned}
\text{WLO}(L;\phi_s) \\
\sim \sum_{\underline{\alpha}\in A} \int_0^\pi dx \Big(\sum_h \exp(i\tfrac{k+2}{2\pi}n(h)\cdot x\tau)\Big) 1_P(x\tau) \\
\times \Big(1_{P\Sigma}(B)\exp\big(\sum_j \epsilon_j \tfrac{1}{2}\big[\alpha_j\big(B(\bar\phi_s(\sigma_j)) + B(\bar\phi_s^{-1}(\sigma_j))\big)\big]\big) \\
\times \prod_{t=1}^\mu \det\big(1_{\mathfrak{g}_0} - \text{Ad}(\exp(B(\sigma_{X_t})))_{|\mathfrak{g}_0}\big)^{\chi(X_t)/2}\Big)_{|B=x\tau+\frac{2\pi}{k+2}\text{ind}(L,\underline{\alpha})} \\
\overset{(+)}{=} \sum_{\underline{\alpha}\in A} \sum_{l=1}^{k+1} \Big(1_{P\Sigma}(B)\exp\big(\sum_j \epsilon_j \tfrac{1}{2}\big[\alpha_j\big(B(\bar\phi_s(\sigma_j)) + B(\bar\phi_s^{-1}(\sigma_j))\big)\big]\big) \\
\times \prod_{t=1}^\mu \det\big(1_{\mathfrak{g}_0} - \text{Ad}(\exp(B(\sigma_{X_t})))_{|\mathfrak{g}_0}\big)^{\chi(X_t)/2}\Big)_{|B=\frac{\pi}{k+2}(l\tau+2\,\text{ind}(L,\underline{\alpha}))}
\end{aligned}
\tag{6.37}
$$

In step (+) we have used that $\{\exp(i\tfrac{k+2}{2\pi}(n(h)\cdot x\tau)) \mid h \in [\Sigma,G/T]\} = \{\exp(i2m(k+2)x) \mid m \in \mathbb{Z}\}$ (cf. Eqs. (6.20) and (6.21)) and as in Subsec. 6.2 above we have again pretended that we can apply the Poisson summation formula. As in Subsec. 6.2 we expect that in a rigorous treatment where, among other things, "loop smearing" and "point smearing" are used in a suitable way this argument can be made rigorous, cf. [26].

For every $l \in \{1,2,\ldots,k+1\}$ and $\underline{\alpha} \in A$ let us now set

$$
\xi_{l,\underline{\alpha}} := l - \sum_{j'} \frac{1}{i}\alpha_{j'}(\tau)\,\text{ind}(l_\Sigma^{j'};\cdot)
\tag{6.38}
$$

Clearly, we can choose $T_1 = \frac{1}{\sqrt{2}}\tau$. Taking into account that $\xi_{l,\underline{\alpha}}$ takes values in $\mathbb{Z}$ and that $\xi_{l,\underline{\alpha}} \cdot \tau = l\tau + 2\,\text{ind}(L,\underline{\alpha})$ we obtain (with the help of Eq. (6.22))

$$
\begin{aligned}
\text{WLO}(L;\phi_s) \sim \sum_{\underline{\alpha}\in A} \sum_{l=1}^{k+1} \Big(1_{\text{Image}(\xi_{l,\underline{\alpha}})\subset\{1,2,\ldots,k+1\}} \prod_{t=1}^\mu \sin(\tfrac{\pi}{k+2}\xi_{l,\underline{\alpha}}(\sigma_{X_t}))^{\chi(X_t)} \\
\times \exp\big(\tfrac{\pi}{k+2}\tfrac{1}{2}\sum_j \alpha_j(\tau)\epsilon_j(\xi_{l,\underline{\alpha}}(\bar\phi_s(\sigma_j)) + \xi_{l,\underline{\alpha}}(\bar\phi_s^{-1}(\sigma_j)))\big)\Big)
\end{aligned}
\tag{6.39}
$$

Taking into account that for sufficiently small $s > 0$ we have $\text{ind}(l_\Sigma^{j'};\bar\phi_s(\sigma_j)) - \text{ind}(l_\Sigma^{j'};\bar\phi_s^{-1}(\sigma_j)) = 0$ if $j \neq j'$, and $\text{ind}(l_\Sigma^{j'};\bar\phi_s(\sigma_j)) - \text{ind}(l_\Sigma^{j'};\bar\phi_s^{-1}(\sigma_j)) \in \{-1,1\}$ (cf. condition (H2) in Subsec. 5.2) and thus $(\text{ind}(l_\Sigma^{j'};\bar\phi_s(\sigma_j)) - \text{ind}(l_\Sigma^{j'};\bar\phi_s^{-1}(\sigma_j)))^2 = 1$ if $j = j'$ we obtain from Eq. (6.38) (for arbitrary l)

$$
\alpha_j(\tau) = -i\big(\text{ind}(l_\Sigma^j;\bar\phi_s(\sigma_j)) - \text{ind}(l_\Sigma^j;\bar\phi_s^{-1}(\sigma_j))\big)\big(\xi_{l,\underline{\alpha}}(\bar\phi_s(\sigma_j)) - \xi_{l,\underline{\alpha}}(\bar\phi_s^{-1}(\sigma_j))\big)
\tag{6.40}
$$

Thus we obtain

$$\text{WLO}(L; \phi_s) \sim \sum_{(l,\underline{\alpha}) \in \text{Pairs}_{adm}} \prod_{t=1}^{\mu} \sin(\tfrac{\pi}{k+2}\xi_{l,\underline{\alpha}}(\sigma_{X_t}))^{\chi(X_t)}$$

$$\times \exp\left(-\tfrac{\pi i}{k+2}\tfrac{1}{2}\sum_j \epsilon_j\left(\text{ind}(l_{\Sigma}^j; \bar{\phi}_s(\sigma_j)) - \text{ind}(l_{\Sigma}^j; \bar{\phi}_s^{-1}(\sigma_j))\right)(\xi_{l,\underline{\alpha}}(\bar{\phi}_s(\sigma_j))^2 - \xi_{l,\underline{\alpha}}(\bar{\phi}_s^{-1}(\sigma_j)))^2)\right) \quad (6.41)$$

where we have set

$$\text{Pairs}_{adm} := \{(l,\underline{\alpha}) \in \{1,\ldots,k+1\} \times A \mid \text{Image}(\xi_{l,\underline{\alpha}}) \subset \{1,2,\ldots,k+1\}\} \quad (6.42)$$

We will now show that the right-hand side of Eq. (6.41) reduces to expression (B.10) in Appendix B (up to a multiplicative constant depending only on the charge k). First we observe that each $(l,\underline{\alpha}) \in \text{Pairs}_{adm}$ determines an area coloring $\eta_{l,\underline{\alpha}}$ of $sh(L)$ with colors in I_{k+2} (cf. Appendix B) given by

$$\eta_{l,\underline{\alpha}}(X_t) = \tfrac{1}{2}(\xi_{l,\underline{\alpha}}(\sigma_{X_t}) - 1) \quad (6.43)$$

with σ_{X_t} as above. It is well-known in the "physical interpretation" of the framework in Appendix B (cf., e.g., [34]) that the color $1/2 \in I_{k+2}$ corresponds to the fundamental representation $\rho_{SU(2)}$ of $SU(2)$. As we have only considered links where all the loops $l_1, l_2, \ldots, l_n$ carry the standard representation $\rho_{SU(2)}$ one should expect that the constant "coloring" $col_{1/2} : \{l_1, l_2, \ldots, l_n\} \to \{0, 1/2, \ldots, k/2\}$ taking only the value $1/2$ will play a role in the sequel. The next proposition (in which we use the notation of Appendix B) shows that this is indeed the case.

Proposition 6.2. *For each $(l,\underline{\alpha}) \in \text{Pairs}_{adm}$ the area coloring $\eta_{l,\underline{\alpha}}$ is admissible w.r.t. $col_{1/2}$ and the mapping $\Xi : \text{Pairs}_{adm} \ni (l,\underline{\alpha}) \mapsto \eta_{l,\underline{\alpha}} \in \text{ad}(sh(L); col_{1/2})$ is a bijection.*

Proof. Ξ is injective: Let us assume without loss of generality that $\sigma_0 \in X_\mu$. Then we have (cf. Eq. (6.25))

$$l = \xi_{l,\underline{\alpha}}(\sigma_0) = 2\eta_{l,\underline{\alpha}}(X_\mu) + 1$$

so l is uniquely determined by $\eta_{l,\underline{\alpha}}$. Moreover, from Eqs. (6.40) and (6.43) it follows that also $\underline{\alpha}$ is uniquely determined by $\eta_{l,\underline{\alpha}}$, so Ξ is injective.

$\Xi(\text{Pairs}_{adm}) \subset \text{ad}(sh(L); col_{1/2})$: Let $(l,\underline{\alpha}) \in \text{Pairs}_{adm}$ and let $e \in E(L)$. As we only consider the special case $DP(L) = \emptyset$ where $E(L) = \{l_\Sigma^1, l_\Sigma^2, \ldots, l_\Sigma^n\}$ we have $e = l_\Sigma^j$ for some fixed $j \leq n$. We have to prove that the triple $(\bar{i}, \bar{j}, \bar{k}) \in I_{k+2}^3$ given by

$$\bar{i} = 1/2, \quad \bar{j} = \eta(X_1(e)), \quad \bar{k} = \eta(X_2(e))$$

fulfills the relations (B.5)–(B.8) in Appendix B with $\bar{r} = k+2$. Here $X_1(e)$ and $X_2(e)$ are defined as in Appendix B. In order to see this first note that for sufficiently small $s > 0$ we have

$$\bar{j} - \bar{k} = \eta(X_1(e)) - \eta(X_2(e))$$
$$= \tfrac{1}{2}\left(\xi_{l,\underline{\alpha}}(\bar{\phi}_s(\sigma_j)) - \xi_{l,\underline{\alpha}}(\bar{\phi}_s^{-1}(\sigma_j))\right)$$
$$= \tfrac{1}{2}\tfrac{\alpha_j(\tau)}{i}\left(\text{ind}(l_\Sigma^j; \bar{\phi}_s(\sigma_j)) - \text{ind}(l_\Sigma^j; \bar{\phi}_s^{-1}(\sigma_j))\right) \quad (6.44)$$

But $\frac{\alpha_j(\tau)}{i} \in \{-1, 1\}$ and $\left(\text{ind}(l_\Sigma^j; \bar{\phi}_s(\sigma_j)) - \text{ind}(l_\Sigma^j; \bar{\phi}_s^{-1}(\sigma_j))\right) \in \{-1, 1\}$ so we obtain

$$|\bar{j} - \bar{k}| = \tfrac{1}{2} \quad (6.45)$$

As $\bar{i} = \tfrac{1}{2}$ this implies relations (B.5) and (B.8). Moreover, Eq. (6.45) implies that at least one of the two numbers $\bar{j}, \bar{k} \in I_{k+2} = \{0, 1/2, 1, \ldots, k/2\}$ must lies even in $\{1/2, 1, \ldots, (k-1)/2\}$. Relations (B.6) and (B.7) now follow easily.

$\Xi(\text{Pairs}_{adm}) \supset \text{ad}(sh(L); col_{1/2})$: Let $\eta \in \text{ad}(sh(L); col_{1/2})$. Let us assume without loss of generality that $\sigma_0 \in X_\mu$. Let $l := 2\eta(X_\mu) + 1$ and let $\alpha_j : \mathfrak{t} \to \mathbb{C}$ be given by

$$\alpha_j(\tau) = -i\,\text{sgn}(X_j^+; l_\Sigma^j))2\left(\eta(X_j^+) - \eta(X_j^-)\right) \quad (6.46)$$

where we have set $X_j^+ := X_1(l_\Sigma^j)$, $X_j^- := X_2(l_\Sigma^j)$ for $l_\Sigma^j \in E(L) = \{l_\Sigma^1, l_\Sigma^2, \ldots, l_\Sigma^n\}$ and where $\mathrm{sgn}(X_j^+; l_\Sigma^j)$ is defined as in Remark B.2. From (B.5)–(B.8) it follows that $l \in \{1, 2 \ldots, k+1\}$ and $\underline{\alpha} := (\alpha_1, \ldots, \alpha_n) \in A$ so $(l, \underline{\alpha}) \in \mathrm{Pairs}_{adm}$. Finally, from Eqs. (6.40), (6.46), (6.43) and

$$\mathrm{sgn}(X_j^\pm; l_\Sigma^j)) = \pm\big(\mathrm{ind}(l_\Sigma^j; \bar{\phi}_s(\sigma_j)) - \mathrm{ind}(l_\Sigma^j; \bar{\phi}_s^{-1}(\sigma_j))\big) \tag{6.47}$$

(which holds if s was chosen sufficiently small) we see that $\eta = \eta_{l,\underline{\alpha}}$ holds. $\qquad\square$

In the sequel we will set $\mathrm{ad}(sh(L)) := \mathrm{ad}(sh(L); col_{1/2})$. Let $X_j^\pm$, $j \leq n$, be defined as in the last part of the proof of Proposition 6.2. Taking into account Eqs. (6.43), (6.47) and Proposition 6.2 we now obtain from Eq. (6.41) (provided that s was chosen sufficiently small)

$$\mathrm{WLO}(L; \phi_s) \tag{6.48}$$

$$\sim \sum_{\eta \in ad(sh(L))} \Big(\prod_{t=1}^\mu \sin(\tfrac{\pi}{k+2}(2\eta(X_t)+1))^{\chi(X_t)}\Big)$$

$$\times \exp\big(-\tfrac{\pi i}{k+2}\tfrac{1}{2}\sum_j \epsilon_j \, \mathrm{sgn}(X_j^+; l_\Sigma^j) \cdot 4((\eta(X_j^+)+1/2)^2 - (\eta(X_j^-)+1/2)^2)\big)$$

$$= \sum_{\eta \in ad(sh(L))} \Big(\prod_{t=1}^\mu \sin(\tfrac{\pi}{k+2}(2\eta(X_t)+1))^{\chi(X_t)}\Big)$$

$$\times \Big\{ \big(\prod_j \exp\big(-\tfrac{\pi i}{k+2}2\epsilon_j \, \mathrm{sgn}(X_j^+; l_\Sigma^j)\big)(\eta(X_j^+)^2 + \eta(X_j^+))$$

$$\times \big(\prod_j \exp\big(-\tfrac{\pi i}{k+2}2\epsilon_j \, \mathrm{sgn}(X_j^-; l_\Sigma^j)\big)(\eta(X_j^-)^2 + \eta(X_j^-))\Big\}$$

$$= \sum_{\eta \in ad(sh(L))} \prod_{t=1}^\mu \Big(\sin(\tfrac{\pi}{k+2}(2\eta(X_t)+1))^{\chi(X_t)} \Big)$$

$$\times \prod_{t=1}^\mu \exp\Big(-\tfrac{\pi i}{k+2}2\Big(\sum_{j \text{ with } arc(l_\Sigma^j) \subset \partial X_t} \epsilon_j \, \mathrm{sgn}(X_t; l_\Sigma^j)\Big)\eta(X_t)(\eta(X_t)+1)\Big)$$

$$= \sum_{\eta \in ad(sh(L))} \prod_{t=1}^\mu \Big(\sin(\tfrac{\pi}{k+2}(2\eta(X_t)+1))^{\chi(X_t)} \times \exp\big(-\tfrac{\pi i}{k+2}2x_t(\eta(X_t) \cdot (\eta(X_t)+1))\big)\Big)$$

$$= \sum_{\eta \in ad(sh(L))} \prod_{t=1}^\mu (v_{\eta(X_t)})^{\chi(X_t)} \sin(\tfrac{\pi}{k+2})^{\chi(X_t)} (-1)^{\chi(X_t)2\eta(X_t)} \exp(2x_t u_{\eta(X_t)})(-1)^{x_t 2\eta(X_t)} \tag{6.49}$$

where u_i, v_j, x_t are given as in Eqs. (B.3), (B.4), and (B.11) in Appendix B.

As each l_Σ^j is – by assumption – a Jordan loop which is 0-homotopic it follows that

$$\chi(X_t) = \#\{j \leq n \mid arc(l_\Sigma^j) \subset \partial X_t\} \qquad \mathrm{mod} \ 2$$

for each $t \leq \mu$. So in the special case where all ϵ_j are odd we have

$$\chi(X_t) = x_t \qquad \mathrm{mod} \ 2$$

for each $t \leq \mu$. If at least one ϵ_j is even then the last equation does not hold in general but using a simple induction over the number of indices j for which ϵ_j is even it follows that one always has

$$\sum_t \chi(X_t)2\eta(X_t) = \sum_t x_t 2\eta(X_t) \qquad \mathrm{mod} \ 2$$

Moreover, we have

$$\prod_t \sin(\tfrac{\pi}{k+2})^{\chi(X_t)} = \sin(\tfrac{\pi}{k+2})^{\chi(\Sigma)} = \sin(\tfrac{\pi}{k+2})^{2-2g}$$

For sufficiently small $s > 0$ we therefore obtain

$$\text{WLO}(L, \phi_s) \sim \sin(\tfrac{\pi}{k+2})^{2-2g} \sum_{\eta \in ad(sh(L))} \prod_{t=1}^{\mu} (v_{\eta(X_t)})^{\chi(X_t)} \exp(2x_t u_{\eta(X_t)}) \tag{6.50}$$

Apart from the constant factor $\sin(\tfrac{\pi}{k+2})^{2-2g}$, which depends only on the charge k but not on the link L, the right-hand side of Eq. (6.50) coincides exactly with the right-hand side of Eq. (B.10) in Appendix B. In particular, $\text{WLO}(L, \phi_s)$ does not depend on the special choice of the points t_0 and σ_0 at the beginning of Sec. 2.

7. Outlook and Conclusions

7.1. **Generalizing the computations of Subsec. 6.3 to links with double points.** In order to complete the computation of the WLOs for $G = SU(2)$ and general links (with standard colors) one has to carry out the following steps:

Firstly, one has to prove that the limits (5.36) exist and one has to calculate their values. Secondly, one has to evaluate expression (5.37) explicitly, for example by rewriting it in terms of "state sums" similar to the ones that appear in Eq. (6.2) in [22]. Finally, one has to perform the $\int \cdots DA_c^{\perp}$ and $\int \cdots DB$ integrations, which can probably be done in a similar way as in Subsec. 6.3. We consider it to be likely that after completing these steps one will finally arrive at an expression for the WLOs that is given by the right-hand side of (B.9).

One word of caution is appropriate here, though: it is not not totally impossible that something similar will happen as in the axial gauge approach to Chern-Simons models on $\mathbb{R}^3$, cf. [19, 4, 21, 22]. In [22] it turned out that, in the Non-Abelian case, the expressions for the WLOs obtained for links with double points depended on the precise way in which the loop smearing regularization procedure was implemented. This "loop smearing dependence" destroys topological invariance and it is thus not surprising that the final expressions obtained for the WLOs in [22] do not fully coincide with the knot polynomial expressions that were expected in the standard literature. (For special non-integer values of the charge k topological invariance could be recovered in [22] within a restricted class of loop smearing procedures, called "axis dependent loop smearing". However, the values of k for which this happens are exactly those that make the relevant knot polynomials trivial).

If one interprets the loop smearing dependence as a reflection of the fact that axial gauge is in a certain sense a rather "singular" gauge then it would seem natural to worry that a similar loop smearing dependence problem (LSD problem) might appear in the torus gauge setting of the present paper when evaluating the WLOs of links with double points. (After all "torus gauge fixing" and "axial gauge fixing" share the aspect of being "singular" gauges).

On the other hand it is clear that the "singularity" of axial gauge fixing alone cannot be "the cause" for the LSD problem. Clearly, axial gauge fixing is equally singular if the structure group G of the model is Abelian but as we saw in [22] some of the additional algebraic relations that hold in the Abelian case prevent the LSD problem from appearing. Instead we prefer to interpret the LSD problem as a reflection of the idea that something is "wrong" with Chern-Simons models on non-compact manifolds. For example, the non-compactness of the manifold $\mathbb{R}^3$ has the unpleasant effect that the expression $S_{CS}(A)$ is not defined for every $A \in \mathcal{A}$. Following [19] we therefore assumed in [22] in several computations that the 1-form A had compact support (or, alternatively, was of rapid decrease). If one could make this assumption consistently then things might not be so bad. One could then try to replace the space $\mathcal{A}$ of all gauge fields on $\mathbb{R}^3$ appearing in the relevant path integral expressions of the form $\int_{\mathcal{A}} \cdots \exp(iS_{CS}(A))DA$ by the space $\mathcal{A}_{comp}$ of 1-forms on $\mathbb{R}^3$ with compact support and hope that the new path integrals reproduce the interesting knot invariants that appeared in [42] for Chern-Simons models on compact manifolds. However, if one wants to apply axial gauge fixing there is (at least) one argument that makes it necessary to work with the original space $\mathcal{A}$ of all gauge fields, cf. the argument in Subsec. 2.2 in [22] that the mapping $\tilde{\mathcal{G}} \times \mathcal{A}^{cax} \ni (\Omega, A) \mapsto A \cdot \Omega \in \mathcal{A}$ is a bijection (here $\mathcal{A}^{cax}$ is the space of 1-forms which are "completely axial" and $\tilde{\mathcal{G}} := \{\Omega \in \mathcal{G} \mid \Omega(0) = 1\}$).

The analogue of this argument where each of the three spaces $\mathcal{A}^{cax}$, $\tilde{\mathcal{G}}$ and $\mathcal{A}$ is replaced by the corresponding subspace of elements with compact support does not hold. In other words: for the approach in [22] it was necessary to "combine" results that hold only for spaces with compact support with results that hold only for the bigger spaces $\mathcal{A}^{cax}$, $\tilde{\mathcal{G}}$ and $\mathcal{A}$. It should therefore not be too surprising that the axial gauge approach for Chern-Simons models on $\mathbb{R}^3$ runs into difficulties (at least when using the implementation of [4, 21, 22]). In fact, the loop smearing dependence was not the only complication/problem in [22]. There were two other problems: Firstly, it turned out in [22] that the values of the WLOs differ from those expected in the standard literature even for the few links for which there was no loop smearing dependence, i.e. for loops without double points[23]. Secondly, in the approach in [4, 21, 22] it was unclear right from the beginning how quantum groups (resp. the corresponding R-matrices) could enter the computations. Note that a quantum group $U_q(\mathfrak{g})$, $q \in U(1)$, is obtained from the classical enveloping algebra $U(\mathfrak{g})$ by a deformation process that involves a fixed Cartan subalgebra $\mathfrak{t}$. But such a Cartan subalgebra never played any role in [4, 21, 22].

By contrast, in the torus gauge approach to Chern-Simons models on $M = \Sigma \times S^1$ a Cartan subalgebra $\mathfrak{t}$ plays an important role right from the beginning. Moreover, as we have seen in Subsec. 6.3 above, in the torus gauge approach to Chern-Simons models on $\Sigma \times S^1$ with *compact*[24] Σ the values of the WLOs of links without double points do agree with those expressions expected in the standard literature. (In [17] it is shown that this is also true for general groups G and general link colorings, cf. point (1) in Subsec. 7.2 below). This makes us optimistic that also the last complication, i.e. the LSD problem will not appear in the torus gauge approach.

Finally, we would like to emphasize that even if it turns out that the LSD problem does appear during the evaluation of the WLOs of general links, the torus gauge approach is still useful:

(a) By studying the WLOs of links consisting exclusively of three vertical loops in the torus gauge approach one obtains a path integral derivation of the Verlinde formula resp. the fusion rules, cf. [10].

(b) It is shown in [17] (cf. point (1) in Subsec. 7.2 below) that by studying WLOs of links that are obtained by taking a loop without double points like in Subsec. 6.3 and adding two vertical loops one can also obtain a path integral derivation of the so-called quantum Racah formula (cf. [37]).

(c) With the help of the torus gauge approach one can probably gain a better understanding of Witten's surgery operations from a path integral point of view. In [42] arguments from Conformal Field Theory are used in order to explain the appearance of the S- and T-matrices in the formulas that relate the values of the WLOs under surgery operations. In [28] we plan to give an alternative explanation which only uses arguments based on the path integral.

7.2. **Other Generalizations/Further Directions.** The generalization of the computations of Subsec. 6.3 to links with double points, discussed in the previous subsection, is clearly the most important open problem that remains to be studied in the torus gauge approach. But there are other directions for a generalization/extension of the results of the present paper which we also find interesting. They are given in the following list:

(1) Generalize the results of Subsec. 6.3 and the results that can be expected if the project described in Subsec. 7.1 can be completed successfully to arbitrary (simple simply-connected compact) groups G and arbitrary link colorings. (Recall that in Subsec. 6.3 we only considered the special situation where $G = SU(2)$ and where all the loops are "colored" with the fundamental representation $\rho_{SU(2)}$). In fact, for the case of links

[23]provided that horizontal framing (="strictly vertical framing" in the terminology of [22]) is used

[24]it is interesting to note that if one evaluates Eq. (3.31) for non-compact Σ, for which the set $[\Sigma, G/T]$ consists of only one element and the summation $\sum_{h \in [\Sigma, G/T]} \cdots$ is therefore trivial, one runs into difficulties. In particular, the values of the WLOs of links without double points are then *not* given by the shadow invariant, which is an other argument in favor of our claim that something is "wrong" with Chern-Simons models on non-compact manifolds

without double points this generalization has already been carried out in [17]. As a by-product of the computations in [17] we obtained a "path integral derivation" of the so-called quantum Racah formula, cf. [37].

(2) In a recent paper, cf. [13], Blau and Thompson study the partition function of Chern-Simons models on 3-manifolds M which are the total spaces of arbitrary S^1-bundles (and not only trivial S^1-bundles as in [10, 11, 12]). Similarly, one can ask whether the results of the present paper can be generalized to Chern-Simons models on arbitrary S^1-bundles.

(3) One can exploit the torus gauge approach for gaining a better understanding of Witten's surgery operations from a path integral point of view, cf. point (c) in the list in Subsec. 7.1 above.

(4) It should be possible to obtain a rigorous realization of the full integral expression on the right-hand side of Eq. (3.31) using results/techniques from white noise analysis, cf. [26]. However, since this treatment based on white noise analysis is rather technical it is natural to look for alternative approaches for making rigorous sense of the right-hand side of Eq. (3.31). For example, one can study approaches which involve a suitable discretization of the base manifold $M = \Sigma \times S^1$. For every fixed triangulation K of Σ and every fixed triangulation of S^1 there is a discrete analogue of the torus gauge fixing procedure and it should be possible to "discretize" the computations in Secs. 5 and 6 of the present paper in such a way that the shadow invariant is recovered within this discretized setting (possibly after taking a suitable continuum limit). Such an approach is currently studied in [27]. We remark that in order to carry out the aforementioned "discretization" successfully, it will probably be necessary to use ideas and constructions from [1, 2] like "field doubling"[25] and the discrete Hodge star operator introduced in Sec. 2 in [2] (see [27] for more details).

7.3. **Conclusions.** In the present paper we have shown how the face models that were introduced in [39] for the definition of the shadow invariant arise naturally when evaluating the right-hand side of Eq. (3.31), which generalizes formula (7.1) in [10]. Although we have carried out all the details only in some special cases we think that it is reasonable to expect (cf. the arguments in Subsec. 7.1 above) that when completing the computations for general links one will finally arrive at the formula (B.9) in Appendix B (or, for $G \neq SU(2)$, at the relevant generalization of formula (B.9) described in [38]). If this turns out to be true then this would mean that we have solved problem (P1) of the introduction for manifolds M of the form $M = \Sigma \times S^1$. Moreover, in view of point (4) in Subsec. 7.2 this would probably also lead to the solution of problem (P2)' for such manifolds.

<div align="center">APPENDIX A: PROOF OF EQ. (5.27)</div>

First we observe that for all $j, j' \in \mathcal{N}$ and $s > 0$ such that

$$\pi_\Sigma(\mathrm{supp}(j)) \cap \pi_\Sigma(\phi_s(\mathrm{supp}(j'))) = \emptyset \tag{A.1}$$

holds, the functions $(\cdot, j)$ and $(\cdot, j')$ on $\mathcal{N}^*$ are independent w.r.t. $\Phi^\perp_{B,\phi_s}$. This follows from the $\Phi^\perp_{B,\phi_s}$-analogues of Eqs. (4.2) and (4.3) (with the help of the polarization identity).

Using the general Wick theorem analogue mentioned in Sec. 4 we see that this statement can be generalized to arbitrary (finite) sequences $j_1, j_2, \ldots, j_m \in \mathcal{N}$, $m \in \mathbb{N}$, such that condition (A.1) holds with $j := j_i$, $j' := j_{i'}$, $i, i' \leq m$. Thus, for small $s > 0$, $m_i \in \mathbb{N}$, $u_1 < u_2 < \ldots < u_{m_i}$, and arbitrary polynomial functions p_i in m_i variables we have: the n-tuple $\psi_1^\epsilon, \ldots, \psi_n^\epsilon$ given by

$$\psi_i^\epsilon = p_i\left(D_{u_1}^{l_i^\epsilon}(\cdot + A_c^\perp + A_{sing}^\perp(\mathrm{h}) + Bdt), \ldots, D_{u_{m_i}}^{l_i^\epsilon}(\cdot + A_c^\perp + A_{sing}^\perp(\mathrm{h}) + Bdt)\right), \quad \epsilon > 0$$

---

[25]involving the dual $\hat{K}$ of K

is independent w.r.t. $\Phi^{\perp}_{B,\phi_s}$ if ϵ is sufficiently small (that the aforementioned support condition is fulfilled for $\psi^{\epsilon}_1, \ldots, \psi^{\epsilon}_n$ and small $\epsilon > 0$ follows from the assumptions that $DP(L) = \emptyset$ and that the framing $(\phi_s)_{s>0}$ is horizontal).

Eq. (5.27) now follows with the help of a suitable limit argument (cf. also Proposition 4 in [22] and the paragraph preceding Eq. (6.3) in [22]).

APPENDIX B: THE SHADOW INVARIANT FOR $M = \Sigma \times S^1$

For the convenience of the reader we will now recall some basic notions from [39], in particular the definition of the "shadow invariant" which was introduced there (cf. also [34]).

For an admissible link L in $M = \Sigma \times S^1$ we will set $D(L) := (DP(L), E(L))$ where $DP(L)$ denotes, as above, the set of double points of L and $E(L)$ the set of curves in Σ into which the loops $l^1_{\Sigma}, l^2_{\Sigma}, \ldots, l^n_{\Sigma}$ are decomposed when being "cut" in the points of $DP(L)$. Clearly, $D(L)$ can be considered to be a finite (multi-)graph. We set

$$\Sigma \backslash D(L) := \Sigma \backslash (\bigcup_j \mathrm{arc}(l^j_{\Sigma})) \tag{B.2}$$

As L was assumed to be admissible (cf. Subsec. 3.1) it follows that the set $\mathcal{C}_{conn}(\Sigma \backslash D(L))$ of connected components of $\Sigma \backslash D(L)$ has only finitely many elements $X_1, X_2, \ldots, X_{\mu}$, $\mu \in \mathbb{N}$, which we will call the "faces" of $\Sigma \backslash D(L)$. In Sect. 3 in [39] it was shown how the link L induces naturally a function $\mathcal{C}_{conn}(\Sigma \backslash D(L)) \to \mathbb{Z}$ which associates to every face $X_t \in \mathcal{C}_{conn}(\Sigma \backslash D(L))$ a number $x_t \in \mathbb{Z}$. x_t was called the "gleam" of X_t and $x'_t := x_t - z_t/2 \in \frac{1}{2}\mathbb{Z}$ with $z_t := \#\{p \in DP(L) \mid p \in \partial X_t\}$ the "modified gleam" of X_t (cf. also Remark e) ii in Sec. 1 of [39]). We will call the pair $sh(L) := (D(L), (x_t)_{t \leq \mu})$ the "shadow" of L.

Let us now fix an $\bar{r} \in \mathbb{N}$ and set

$$I := I_{\bar{r}} := \{0, 1/2, 1, 3/2, \ldots, (\bar{r} - 2)/2\}$$

For each $j \in I$ we set

$$u_j := \pi i(j - j(j+1)/\bar{r}) = \pi ij - \tfrac{\pi i}{\bar{r}}j(j+1), \tag{B.3}$$

$$v_j := (-1)^{2j} \frac{\sin((2j+1)\pi/\bar{r})}{\sin(\pi/\bar{r})} \tag{B.4}$$

A "coloring" of L with colors in I is a mapping $col : \{l_1, l_2, \ldots, l_n\} \to I$. An "area coloring" of $sh(L)$ with colors in I is a mapping $\eta : \{X_1, \ldots, X_{\mu}\} \to I$. In the sequel let us fix a coloring col of L. Clearly, col induces a mapping $E(L) \to I$, which will also be denoted by col. For every $e \in E(L)$ let $X_1(e)$ and $X_2(e)$ denote the two faces that are "touched" by e. More
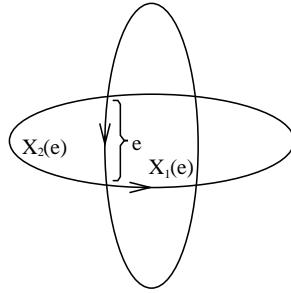

FIGURE 1.

precisely, $X_1(e)$ (resp. $X_2(e)$) denotes the unique face X_t such that $e \subset \partial X_t$ and, additionally, the orientation which is induced on e by the orientation on ∂X_t coincides with (resp. is opposite to) the orientation which e inherits from the loop on which it lies (this is illustrated in Figure 1 for a graph with 4 vertices and 8 edges).

An area coloring η will be called "admissible" w.r.t. *col* if for all $e \in E(L)$ the triple $(\bar{i}, \bar{j}, \bar{k})$ given by

$$\bar{i} = col(e), \quad \bar{j} = \eta(X_1(e)), \quad \bar{k} = \eta(X_2(e))$$

fulfills the relations

$$\bar{i} + \bar{j} + \bar{k} \in \mathbb{Z} \tag{B.5}$$

$$\bar{i} + \bar{j} + \bar{k} \leq \bar{r} - 2 \tag{B.6}$$

$$\bar{i} \leq \bar{j} + \bar{k} \tag{B.7}$$

$$\bar{j} \leq \bar{k} + \bar{i}, \quad \bar{k} \leq \bar{i} + \bar{j} \tag{B.8}$$

The set of all admissible area colorings η of $sh(L)$ w.r.t. *col* will be denoted by $\mathrm{ad}(sh(L); col)$ or simply by $\mathrm{ad}(sh(L))$.

Note that every pair $(p, \eta) \in DP(L) \times \mathrm{ad}(sh(L); col)$ induces a 6-tuple $(\bar{i}, \bar{j}, \bar{k}, \bar{l}, \bar{m}, \bar{n}) \in I^6$ given by

$$\bar{i} = col(e_1(p)), \quad \bar{l} = col(e_2(p))$$

and

$$\bar{j} = \eta(X_1(p)), \quad \bar{k} = \eta(X_2(p)), \quad \bar{m} = \eta(X_3(p)), \quad \bar{n} = \eta(X_4(p))$$

where $e_1(p)$ and $e_2(p)$ are the two edges "starting" in p and $X_1(p), X_2(p), X_3(p), X_4(p)$ the four faces that "touch" the point p, cf. Figure 2. We can now define the "shadow invariant" $|\cdot|$ by
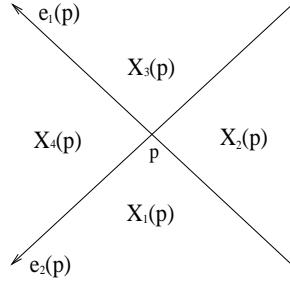

FIGURE 2.

$$|sh(L)| = \sum_{\eta \in ad(sh(L))} \left(\prod_{p \in DP(L)} \mathrm{symb}_q(\eta, p) \right) \left(\prod_{t=1}^{\mu} (v_{\eta(X_t)})^{\chi(X_t)} \exp(2x'_t u_{\eta(X_t)}) \right) \tag{B.9}$$

for all L as above. Here $\mathrm{symb}_q(\eta, p)$ denotes the so-called quantum 6j-symbol which is associated to the number $q := \exp(\frac{2\pi i}{\bar{r}}) \in \mathbb{C}$ and to the 6-tuple $(\bar{i}, \bar{j}, \bar{k}, \bar{l}, \bar{m}, \bar{n})$ induced by (η, p) (for more details, see [39]).

Remark B.1. Actually, $|\cdot|$ was defined in [39] as a function on the set of *framed* shlinks. What we denote by $|sh(L)|$ is in fact the (value of the) shadow invariant for the framed shlink $sh(L)$ which is obtained from L when equipping L with a "vertical" framing in the sense of [39]. Note that in the special case where the link L has no double points "vertical" framing in the sense of [39] is equivalent to what we called "horizontal" framing in Subsec. 5.2 above.

Remark B.2. In the special case where the link L has no double points, i.e. $DP(L) = \emptyset$, we have $E(L) = \{l_\Sigma^1, \ldots, l_\Sigma^n\}$ and $x'_t = x_t$ so formula (B.9) reduces to

$$|sh(L)| = \sum_{\eta \in ad(sh(L))} \prod_{t=1}^{\mu} (v_{\eta(X_t)})^{\chi(X_t)} \exp(2x_t u_{\eta(X_t)}) \tag{B.10}$$

Moreover, in this special case x_t is simply given by

$$x_t = \sum_{j \text{ with } \mathrm{arc}(l_\Sigma^j) \subset \partial X_t} \epsilon_j \cdot \mathrm{sgn}(X_t; l_\Sigma^j) \tag{B.11}$$

where $\epsilon_j = \text{wind}(l_{S^1}^j)$ and where we have set $\text{sgn}(X_t; l_\Sigma^j) := 1$ (resp. $\text{sgn}(X_t; l_\Sigma^j) := -1$) if the orientation on $\partial X_t = \text{arc}(l_\Sigma^j)$ which is induced by the orientation on X_t coincides with (resp. is opposite to) the orientation that is induced by $l_\Sigma^j : S^1 \to \text{arc}(l_\Sigma^j)$.

References

[1] D. H. Adams. A doubled discretization of abelian Chern-Simons theory. *Phys. Rev. Lett.*, 78(22):4155–4158, 1997.

[2] D. H. Adams. R-Torsion and Linking Numbers from Simplicial Abelian Gauge Theories. Preprint, arXiv:hep-th/9612009

[3] S. Albeverio and J. Schäfer. Abelian Chern-Simons theory and linking numbers via oscillatory integrals. *J. Math. Phys.*, 36(5):2135–2169, 1994.

[4] S. Albeverio and A.N. Sengupta. A Mathematical Construction of the Non-Abelian Chern-Simons Functional Integral. *Commun. Math. Phys.*, 186:563–579, 1997.

[5] D. Altschuler and L. Freidel. Vassiliev knot invariants and Chern-Simons perturbation theory to all orders. *Comm. Math. Phys.*, 187:261–287, 1997.

[6] S. Axelrod and I.M. Singer. Chern-Simons perturbation theory. In Catto, Sultan et al., editor, *Differential geometric methods in theoretical physics. Proceedings of the 20th international conference, June 3-7, 1991, New York City, NY, USA*, volume 1-2, pages 3–45. World Scientific, Singapore, 1992.

[7] S. Axelrod and I.M. Singer. Chern-Simons perturbation theory. II. *J. Differ. Geom.*, 39(1):173–213, 1994.

[8] D. Bar-Natan. On the Vassiliev knot invariants. *Topology*, 34:423–472, 1995.

[9] D. Bar-Natan. Perturbative Chern-Simons theory. *J. Knot Theory and its Ramifications*, 4:503–547, 1995.

[10] M. Blau and G. Thompson. Derivation of the Verlinde Formula from Chern-Simons Theory and the G/G model. *Nucl. Phys.*, B408(1):345–390, 1993.

[11] M. Blau and G. Thompson. Lectures on 2d Gauge Theories: Topological Aspects and Path Integral Techniques. In E. Gava et al., editor, *Proceedings of the 1993 Trieste Summer School on High Energy Physics and Cosmology*, pages 175–244. World Scientific, Singapore, 1994.

[12] M. Blau and G. Thompson. On Diagonalization in $Map(M,G)$. *Commun. Math. Phys.*, 171:639–660, 1995.

[13] M. Blau and G. Thompson. Chern-Simons Theory on S^1-Bundles: Abelianisation and q-deformed Yang-Mills Theory. *J. High Energy Phys.*, 5:3–37, 2006 (electronic).

[14] R. Bott and C. Taubes. On the self-linking of knots. *J. Math. Phys.*, 35(10):5247–5287, 1994.

[15] Th. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

[16] S. de Haro. A Note on Knot Invariants and q-Deformed 2d Yang-Mills, arXiv:hep-th/0509167, 2005.

[17] S. de Haro and A. Hahn. The Chern-Simons path integral and the quantum Racah formula, Preprint, arXiv:math-phys/0611084v1

[18] D. S. Freed. Quantum groups from path integrals. q-alg/9501025 Preprint.

[19] J. Fröhlich and C. King. The Chern-Simons Theory and Knot Polynomials. *Commun. Math. Phys.*, 126:167–199, 1989.

[20] E. Guadagnini, M. Martellini, and M. Mintchev. Wilson Lines in Chern-Simons theory and Link invariants. *Nucl. Phys. B*, 330:575–607, 1990.

[21] A. Hahn. Chern-Simons theory on $\mathbb{R}^3$ in axial gauge: a rigorous approach. *J. Funct. Anal.*, 211(2):483–507, 2004.

[22] A. Hahn. The Wilson loop observables of Chern-Simons theory on $\mathbb{R}^3$ in axial gauge. *Commun. Math. Phys.*, 248(3):467–499, 2004.

[23] A. Hahn. Chern-Simons models on $S^2 \times S^1$, torus gauge fixing, and link invariants I. *J. Geom. Phys.*, 53(3):275–314, 2005.

[24] A. Hahn. Chern-Simons models on $S^2 \times S^1$, torus gauge fixing, and link invariants II. *J. Geom. Phys.*, 58:1124–1136, 2008.

[25] A. Hahn. A path integral approach to quantum groups, work in progress.

[26] A. Hahn. White noise analysis in the theory of three-manifold quantum invariants. In A.N. Sengupta and P. Sundar, editors, *Infinite Dimensional Stochastic Analysis*, volume XXII of *Quantum Probability and White Noise Analysis*, pages 201–225. World Scientific, Singapore, 2008.

[27] A. Hahn. From simplicial Chern-Simons theory to the shadow invariant I, work in progress.

[28] A. Hahn. Surgery operations on the Chern-Simons path integral from a conditional expectations point of view, work in progress

[29] S.-T. Hu. *Homotopy Theory*. Academic Press, New York and London, 1959.

[30] L. Kauffman. *Knots*. Singapore: World Scientific, 1993.

[31] L. Kauffman. Functional integration, Kontsevich integral and formal integration. *J. Korean Math. Soc.*, 38(2):437–468, 2001.

[32] G. Kuperberg. Quantum invariants of knots and 3-manifolds (book review). *Bull. Amer. Math. Soc.*, 33(1):107–110, 1996.

[33] P. Leukert and J. Schäfer. A Rigorous Construction of Abelian Chern-Simons Path Integrals using White Noise Analysis. *Rev. Math. Phys.*, 8(3):445–456, 1996.

[34] M. Polyak and N. Reshetikhin. On 2D Yang-Mills Theory and Invariants of Links. In Sternheimer D. et al, editor, *Deformation Theory and Symplectic Geometry*, pages 223–246. Kluwer Academic Publishers, 1997.

[35] N.Y. Reshetikhin and V.G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Commun. Math. Phys.*, 127:1–26, 1990.

[36] N.Y. Reshetikhin and V.G. Turaev. Invariants of three manifolds via link polynomials and quantum groups. *Invent. Math.*, 103:547–597, 1991.

[37] S. Sawin. Quantum groups at roots of unity and modularity. *J. Knot Theory Ramifications*, 15(10):1245–1277, 2006.

[38] V. G. Turaev. *Quantum invariants of knots and 3-manifolds.* de Gruyter, 1994.

[39] V.G. Turaev. Shadow links and face models of statistical mechanics. *J. Diff. Geom.*, 36:35–74, 1992.

[40] V.G. Turaev and O. G. Viro. State sum invariants of 3-manifolds and quantum 6j-symbols. *Topology*, 31(4):865–902, 1992.

[41] E. Witten. Gauge theories and integrable lattice models. *Nuclear Phys. B*, 322(3):629–697, 1989.

[42] E. Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121:351–399, 1989.

[43] E. Witten. Gauge theories, vertex models, and quantum groups. *Nuclear Phys. B*, 330(2-3):285–346, 1990.